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On a Flow Reducing Volume Within Hamiltonian Isotopy Class

Jingyi Chen*

Department of Mathematics, the University of British Columbia, Vancouver, B.C. V6T 1Z2, Canada.

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To Professor Gang Tian in the occasion of his 65th birthday.

Abstract. We survey basic properties of the geometric flow for immersions within a Hamiltonian isotopy class and propose a definition for Type I singularities.

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1 Introduction

Let M be a symplectic manifold of dimension $2n$ with a symplectic form ω and let J and h be an almost complex structure and h a Riemannian metric on M satisfying $\omega(X,Y)=h(JX,Y)$ for all $X,Y\in T_pM$ for any $p\in M$. We say M is almost Kähler in this situation. For a family of immersions

$$F:L\times[0,T)\to M$$

of an n-dimensional manifold L into M, we consider the evolution

$$\frac{\partial F}{\partial t}=J\nabla \mathrm{div}(JH). \tag{1.1}$$

where H is the mean curvature vector of F, div and ∇ are along L in the induced metric from h, and tangential diffeomorphisms are taken such that $\partial F/\partial t$ is normal to L everywhere.

It is shown in [5] that the flow exists for shorttime uniquely, stays in its initial Hamiltonian isotopy class while reduces volume during the evolution, and extends smoothly to a longer time as long as the length of the second fundamental form $|A|$ remains bounded.

In this note, we provide some further observations on this flow.

*Corresponding author. *Email addresses:* `jychen@math.ubc.ca` (Chen J)

2 Degeneracy in parabolicity

Unlike many geometric flows, for example, the mean curvature flow and the Willmore flow, the shorttime existence of (1.1) when $\dim L > 1$ does not follow from the standard theory for parabolic systems due to the fact that degeneracy of parabolicity is not caused just by tangential directions. We demonstrate this by a computation of the principal symbol of the fourth order operator on the right hand side of (1.1) (Appendix A in [5]).

Let (M,J,g,ω) be almost Kähler. Consider the operator

$$P = J\nabla \mathrm{div}(JH)$$

acting on the set of immersions

$$\mathcal{F}(L) = \left\{ F : L^n \to M^{2n} \text{ is a smooth immersion} \right\}.$$

It is known ([12]) the symbol of the linearized operator of H is

$$L_H(X,\xi) = |\xi|^2 \pi^\perp(X)$$

where $\pi^\perp : TM \to T^\perp F(L)$ is the projection to the normal bundle, $X \in TM$ and $0 \neq \xi \in T^*M$ (more precisely, at any point of the submanifold $F(L)$).

For a fixed F, we consider the second order differential operator

$$Q = J\nabla \mathrm{div}_F J : C^\infty\left(L, F^* T^\perp F(L)\right) \to C^\infty(L, F^* TM)$$

between smooth sections of the pullback vector bundles. We compute its linearization at a normal vector ν_0. Let $x^i, i = 1,\cdots,n$ be coordinates around some point $p \in L$ and $y^a, \alpha = 1,\cdots,2n$ be coordinates around $F(p)$ on M. Denote

$$F_i = \frac{\partial F^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha} \in TF(L)$$

for the tangent vectors and

$$g_{ij} = g\left(F_i^\alpha \frac{\partial}{\partial y^\alpha}, F_j^\beta \frac{\partial}{\partial y^\beta} \right) = F_i^\alpha F_j^\beta g_{\alpha\beta}$$

for the induced metric on $F(L)$.

To investigate the initial value problem of the flow, we are interested in the behaviour of the operator P in a neighbourhood of the initial Lagrangian submanifold. For this purpose, it is natural to consider totally real immersions (an open condition) rather than Lagrangian immersions (a closed condition). The reason for this is that, except the initial immersion, the immersions are not known to be Lagrangian before establishing the existence of the flow.