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Quantitative Uniqueness for Mean Curvature Flow

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To our friend Gang Tian.

Abstract. We show that by applying a set of existing analytical arguments, a more robust effective uniqueness result for blowups can be obtained, with multiple implications following therefrom.

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1 Introduction

There is a natural scaling for a mean curvature flow (MCF) $M_s \subset \mathbf{R}^{n+1}$, where space and time dilate parabolically. A general limit flow at a space-time point $(\bar{x},\bar{s})$ is a limit of a sequence of rescalings $\frac{1}{\mu_i}\left(M_{s_i+\mu_i^2 s}-x_i\right)$ centered at a sequence of points $(x_i,s_i)\to(\bar{x},\bar{s})$ with $\mu_i\to 0$. Typically, the time-slices of the limit are non-compact and the convergence is on compact sets. When the dilations are all centered at the same point, then the limit flow is called a tangent flow.

A tangent flow at the origin in space-time is the limit of a sequence of rescaled flows $\frac{1}{\delta_i}M_{\delta_i^2 t}$ where $\delta_i\to 0$. By a monotonicity formula of Huisken [17], and an argument of Ilmanen and White [18,23], tangent flows are shrinkers, i.e., self-similar solutions of MCF that evolve by rescaling. A priori, different sequences δ_i could give different tangent flows and the question of the uniqueness of the blowup is whether it is independent of the sequence. Uniqueness has strong implications for regularity [9,12,13], cf. [24,25].

A singular point is *cylindrical* if some tangent flow is a multiplicity one cylinder $\mathbf{S}^k\times\mathbf{R}^{n-k}$; [7] proved that cylindrical blowups are unique. The main tool was a Lojasiewicz-type inequality that led to a rate of decay for the gaussian area and a rate of convergence

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to the limit. This was inspired by the way that Lojasiewicz proved uniqueness for finite dimensional analytic gradient flows [19]. The formal similarities to [19] helped frame the problem. However, those methods did not apply for a number of reasons, including the non-compactness of the time-slices. Instead new ideas and techniques were required.

The next theorem will illustrate why effective uniqueness is useful. Suppose that for $s\in[-1,1]$, $\lambda(M_s)\leq\lambda_0$ (where λ is defined in (1.2)) and M_s satisfies:

(A) The origin $(0,0)$ is a cylindrical singularity with blow up $\mathcal{C}=\mathbf{S}^k_{\sqrt{2k}}\times\mathbf{R}^{n-k}$.

(B) There are sequences $x_i\to 0$, $s_i\to 0$ and $\mu_i\to 0$ so that

$$\frac{1}{\mu_i}\left(M_{s_i-\mu_i^2}-x_i\right)$$

converges smoothly with multiplicity one to $\mathcal{O}(\mathcal{C})$, where $\mathcal{O}$ is a rotation in $\mathbf{R}^{n+1}$.

We will see that $\mathcal{O}(\mathcal{C})=\mathcal{C}$, i.e., the two cylinders are the same:

Theorem 1.1. *If (A) and (B) hold, then* $\mathcal{O}(\mathcal{C})=\mathcal{C}$.

If the sequence in (B) was centered at the origin in space-time (i.e., if $x_i=0$ and $s_i=0$), then this is uniqueness of cylindrical blow ups from [7]. The more general case, where the centers of the rescalings converge to the origin, will follow from an effective uniqueness theorem; see Theorem 1.2 below.

It is cleanest to state the effective uniqueness for solutions of the rescaled MCF. The rescaled MCF is the gradient flow for the F-functional or Gaussian surface area

$$F(\Sigma)=(4\pi)^{-n/2}\int_\Sigma \mathrm{e}^{-\frac{|x|^2}{4}}\,d\mu. \tag{1.1}$$

The entropy [6], is the supremum of the Gaussian surface areas over all centers and scales

$$\lambda(\Sigma)=\sup_{c>0,x_0\in\mathbf{R}^{n+1}} F(x_0+c\Sigma). \tag{1.2}$$

Definition 1.1. *We will say that* $dist_R(\Sigma,\Gamma)<\epsilon$ *if* $B_R\cap\Sigma$ *can be written as a* $C^{2,\alpha}$ *graph over (a subset of)* Γ *of a function with* $C^{2,\alpha}$ *norm less than* ϵ.

The next theorem shows that if a rescaled MCF starts off close to a cylinder and F does not decrease much, then the flow does not change much. The key point is that this is independent of the time flowed. In the theorem, Σ_t is an n-dimensional rescaled MCF with entropy $\lambda(\Sigma_t)\leq\lambda_0$ and all constants are allowed to depend on n and λ_0.

Theorem 1.2. *There exist* $c,\alpha,\epsilon_1,\epsilon_2>0$ *and* $R_1,R_2>2n$ *so that if* Σ_t *is defined on* $[t_1,t_2]$,

1. $dist_{R_1}(\Sigma_t,\mathcal{C})<\epsilon_1$ *for* $t\in[t_1,t_1+2]$, *and*