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Zero Set of the Solution to One-Dimensional Parabolic Equation

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To Professor Gang Tian in the occasion of his 65th birthday.

Abstract. In this paper, we investigate the one-dimensional parabolic equation. Using blow-up analysis, we prove that the zero set at each time slice is discrete and that the number of zeros decreases with respect to time. Moreover, it strictly declines at the singularity. This provides a new and simplified proof of the main results in existing literature, where the original proof relies on spectral theory. Besides, our results represent a localized version of those existing results with weaker conditions.

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Key words: Nodal set, singular set, parabolic equations, blow-up analysis.

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1 Introduction

In this paper, we study the zero set of a nonzero solution $u(x,t)$ of the following parabolic equation

$$\partial_t u = a(x,t)\partial_{xx}u + b(x,t)\partial_x u + c(x,t)u \tag{1.1}$$

defined in $\Omega\times I$, where both Ω and I are intervals in $\mathbb{R}$.

The results of our study can be interpreted as an extension of the Sturm theorems, see [6]. Specifically, considering the following Dirichlet problem:

$$\begin{cases} \partial_t u = \partial_{xx}u + q(x)u, & x\in[0,2\pi], t\in(0,\infty); \\ u(0,t)=u(2\pi,t)=0, & t\in(0,\infty); \\ u(x,0)=\phi(x), & x\in[0,2\pi]. \end{cases}$$

where $\phi(x)$ is a given smooth function. Let $N(t)$ denote the number of zeros of $u(x,t)$ within $\Omega=(0,2\pi)$. The Sturm theorems imply that $N(t)$ is a nonincreasing function with

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respect to t. Moreover, in the general case, where a,b and c are smooth functions and $a \geq a_0 > 0$, the above Sturm's theorems remain valid.

Sturm's theorems are extensively used in the analysis of the dynamics of semilinear parabolic equations, however, it does not exclude the possibility of $N(t) = \infty$. In 1988 Angenent [2] asserts that $N(t) < \infty$ within a bounded interval. More geometrically, the zero curves either merge or vanish after the singularity. The applications of this result has covered many fields, including the curve shortening problem, the classification of ancient solutions to the curve shortening flow, diffusion equations, mean curvature flow, and low-entropy flow, among others. We refer readers to [1,3,5,7,8].

The proof of Angenent involves spectral theory by analyzing the properties of the operator $Lu = \partial_{xx}u + q(x)u$ on the whole line $\mathbb{R}$. In this paper, we use blow-up analysis to show a local version of the results of Angenent in 1988. Now we assume that $a > 0$,

$$a, a^{-1}, a_x, a_t, b \qquad \text{and} \qquad c \in L^\infty. \tag{1.2}$$

Denote the t-slice zero set and singular set of u as follows

$$\begin{aligned} &Z_t = \{x \in \Omega : u(x,t) = 0\}, \\ &S_t = \{x \in \Omega : u(x,t) = \partial_x u(x,t) = 0\}, \\ &S_u = \cup_{t \in I} S_t \times \{t\} = \{(x,t) \in \Omega \times I : u(x,t) = \partial_x u(x,t) = 0\}. \end{aligned} \tag{1.3}$$

Here Z_t, S_t is defined on space, and the singular set is defined without $\partial_t u = 0$. We first consider $\Omega = (-1,1)$ and $I = (-1,1)$, and the following is the first result in this paper.

Theorem 1.1. *Let $u : \Omega \times I \to \mathbb{R}$ be a solution to* (1.1) *with coefficients conditions* (1.2). *Assume that $u(x,0)$ is not identical to zero in Ω. If $0 \in Z_0$, then it is an isolated point in Z_0. Furthermore, if $0 \in S_0$, then there exist positive constants δ and ϵ such that the following statements about zero set hold:*

(1) For any $t \in (-\delta, 0)$, $|Z_t \cap (-\epsilon, \epsilon)| \geq 2$;

(2) For any $t \in (0, \delta)$, $|Z_t \cap (-\epsilon, \epsilon)| \leq 1$.

Where $|A|$ denotes the number of elements in A.

Next we consider $\Omega = \mathbb{R}$ and $I = (0,T)$, and assume the growth condition

$$|u(x,t)| \leq K_1 e^{K_2 |x|^2} \tag{1.4}$$

for some positive constants K_1, K_2. Noting that Theorem 1.1 is a local result, we can use similar arguments to show that the above results still hold for parabolic equation defined on a small rectangle. As a consequence of Theorem 1.1, we have the following theorem.

Theorem 1.2. *Let $u : \mathbb{R} \times (0,T) \to \mathbb{R}$ be a nonzero solution to* (1.1) *with conditions* (1.2) *and* (1.4). *Then the following statements about zero set hold:*