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Some Remarks on Fibrations in Complex Geometry

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To Gang Tian on the occasion of his 65th birthday.

Abstract. In this article, we discuss some properties of holomorphic fibrations in the complex analytic setting.

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1 Introduction

In this article, we will discuss some general properties of holomorphic fibrations and give some applications. We will also prove a naturality property of the Leray spectral sequence with respect to open inclusions.

1.1 Holomorphic fibrations

The main objects of interest in this article are the following.

Definition 1.1. *Let Z and Y be reduced and irreducible complex spaces satisfying* $1 \leq \dim(Y) < \dim(Z)$. *A fibration $f: Z \to Y$ is a proper surjective holomorphic mapping with connected fibers. We say that Z fibers over Y and call Y the base of the fibration. The fiber of f over a point y is the set $f^{-1}(y)$ with the fiber product structure, and is denoted by $Z_y = Z \times_Y spec(k(y))$.*

In Section 2, we will discuss some general properties of fibrations, such as fiber dimension, discriminant locus, monodromy, ordinary points, and non-ordinary points. Next, in Section 3, we will discuss the case where the base Y is a curve C. In this case,

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if Z is smooth and compact, then the discriminant locus is a finite set, so there are only finitely many singular fibers. For any topological space X, we let $\chi(X)$ denote the topological Euler characteristic. The monodromy is defined in Subsection 2.3. We will prove the following estimate on the first Betti number $b^1(Z_y)$ of the singular fibers.

Theorem 1.1. *Let $f:Z\to C$ be a fibration, where Z is a compact complex n-manifold and C is a smooth curve. Then*

$$b^1(Z_y)\leq b^1(F) \tag{1.1}$$

for all $y\in C$, where F is the general fiber. Equality holds in (1.1) *for some $y\in C$ if and only if the local monodromy $\rho_y^{(1)}$ around y is trivial. If $n=2$, then we have the inequality*

$$\chi(Z)\geq\chi(F)\chi(C), \tag{1.2}$$

with equality if and only if the global monodromy $\rho^{(1)}$ is trivial and every fiber of f is irreducible.

For $n=2$, this was previously proved in [2, Chapter III.11], but the higher-dimensional case of (1.1) seems new. We note that if Z were assumed to be Kähler, then the local invariant cycles theorem is true in higher dimensions, in which case Theorem 1.1 is known; see [7]. In Subsection 3.3, we will also give some application to a local invariant cycles theorem in the non-Kähler case; see Corollary 3.1.

In Section 4, we will discuss some properties of normal surface singularities. In particular, we will define the notion of a rational tree singularity, which is generalization of a rational singularity; see Definition 4.2.

We next turn our attention to the case where Z is a threefold and Y is a surface. In Section 4, we will prove the following result in the case Z is a conic bundle over a surface with rational tree singularities.

Theorem 1.2. *Let $f:Z\to Y$ be a fibration, where Z is a compact connected threefold and Y^2 is a compact surface with rational tree singularities. Assume that the generic fiber F of f is a smooth rational curve. Then $b^1(Z_y)=0$ for all $y\in Y$. If in addition the fibers of f are equidimensional, then*

$$\chi(Z)\geq 2\cdot\chi(Y), \tag{1.3}$$

with equality if and only if every fiber is homeomorphic to $\mathbb{P}^1$.

In the case of equality in (1.3) it is tempting to speculate that every fiber would have to be a smooth $\mathbb{P}^1$, but we are unable to make such a strong conclusion with our methods.

Remark 1.1. If Z and Y were assumed to be projective with Y only having rational singularities, then this result can be seen to follow from [27, Theorem 7.1] (where even stronger results are obtained). We emphasize that in Theorem 1.2, we do not make any projectivity or Kählerian assumption.

The main tool in the proof of Theorem 1.2 is a certain naturality property of the Leray spectral sequence, which we discuss next.