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Group-Invariant Solutions and Conservation Laws of One-Dimensional Nonlinear Wave Equation

Ben Yang[1], Yunjia Song[1], Yanzhi Ma[1] and Xinxue Zhang[1,†]

Abstract Based on classical Lie symmetry method, the one-dimensional nonlinear wave equation is investigated. By using four-dimensional subalgebras of the equation, the invariant groups and commutator table are constructed. Furthermore, optimal system of the equation is obtained, and the exact solutions can be gained by solving reduced equations. Finally, a complete derivation of the conservation law is given by using conservation multipliers.

Keywords One-dimensional nonlinear wave equation, Lie symmetry, optimal system, conservation law

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1. Introduction

Wave equations describe various wave phenomena and have a wide range of applications in the fields of physics [19], biology and engineering [22,25], making the solution of wave equations indispensable. The methods of solving partial differential equations (PDEs) mainly include (G'/G) expansion method [6,14,35], extended hyperbolic method [26], inverse scattering method [7], exponential function method [32], generalized exp-function method [13], Bäcklund transformation method [33], Jacobi elliptic method [2,4,15], hyperbolic tangent method [1], F-expansion method [11], homogeneous equilibrium method [24], Lie symmetry analysis method [12,20,21,29] and so on [3,9,10,34].

The Lie symmetry method can solve PDEs efficiently. In this article, we consider a one-dimensional nonlinear wave equation

$$u_{tt} = \left((1+u)^{2a} u_x\right)_x, \tag{1.1}$$

where u is a function of x, t and $a > 0$ is a constant. In [5], Ames, Lohner and Adams proposed a general nonlinear fluctuation equation

$$u_{tt} = [\mathscr{B}(u)u_x]_x, \tag{1.2}$$

where $\mathscr{B}$ is expressed as a function of u. Then, they discussed equation (1.2) with Lie symmetric analysis, and derived explicit invariant solutions to wave propagation and

†The corresponding author.
Email address: yangben09@163.com (B. Yang), zhangxinxue01@163.com (X. Zhang)
1School of Mathematical Sciences, Liaocheng University, Liaocheng, Shandong 252000, China

transonic equation in gases. Furthermore, Sophocleous and Kingston [27] attempted the following three special cases

$$u_{tt} = F(u_x)\, u_{xx}, \quad u_{tt} = F(u_{xx}), \quad u_{xt} = F(u). \tag{1.3}$$

Equation (1.3) exists in the discrete symmetries groups which form finite order cyclic groups. In [16], Hu studied the degenerate initial-boundary value problem of equation (1.1), and obtained the global existence of the solution by using the eigendecomposition method under relaxed conditions. The global existence of the solution to a more general 2×2 conservation system of equation (1.1) was proven in [18, 30]. In [8, 31], the conservation system of equation (1.1) has been studied. In [28], Sugiyama introduced the large-time behavior of the solution of the equation under the Cauchy problem, obtained the sufficient conditions for the degradation of the equation in finite time and derived a threshold for the global existence and degradation of the separated solution.

This article mainly includes the following sections. In the second section, the concepts of Lie symmetry and prolongation method are introduced, followed by a study on Lie point symmetry and one-dimensional optimal system of the equation. Investigating the group invariant solutions of the equation by the optimal system, we obtain the exact solutions through symmetry reduction. The third section discusses the conservation laws of the equation. Finally, a simple summary is drawn.

2. Lie symmetry analysis and optimal system of equation (1.1)

2.1. Definition introduction

Based on the conclusions of Sophus Lie, some concepts of Lie symmetry [23] have been set up.

Assume that the s-order partial differential equation system Q with q independent variable and m dependent variable is

$$\Delta q\left(x, u^{(n)}\right) = 0\ , \quad Q = 1, 2, 3, \cdots, k, \tag{2.1}$$

in which $x = \left(x^1, x^2, \cdots, x^q\right)$, $u = \left(u^1, u^2, \cdots, u^m\right)$ and $u^{(n)}$ represents arbitrary order derivative of u, and its range of value is from 0 to n. Now, let us discuss the infinitesimal one-parameter Lie group transformation of the system

$$\bar{x}^k = x^k + \varepsilon \xi^k(x, u) + o\left(\varepsilon^2\right), \quad \bar{u}^p = u^p + \varepsilon \phi^p(x, u) + o\left(\varepsilon^2\right), \tag{2.2}$$

where ε is an arbitrary, and ξ^k, ϕ^p represent the infinitesimal transformations of function independent variables and dependent variables respectively.

Considering the n-order differential equations for u

$$\Delta\left(x, u^{(n)}\right) = 0, \tag{2.3}$$

in which Δ denotes a smooth mapping from $X \times U^{(n)}$ to $\mathbb{R}$:

$$\Delta : X \times U^{(n)} \to \mathbb{R}.$$

The following subset can be obtained by equation (2.3)

$$S' = \left\{(x, u^n) : \Delta\left(x, u^{(n)}\right) = 0\right\} \subset X \times U^{(n)}.$$

Assume that S' is an open subset of $X \times U^{(2)}$, and $\Delta\left(x, u^{(2)}\right) = u_{tt} - \left((1+u)^{2a} u_x\right)_x = 0$ is the n-order equation defined on S'. Then, the vector v on the open subset S' is

$$v = \xi\left(x, t, u\right) \frac{\partial}{\partial x} + \tau\left(x, t, u\right) \frac{\partial}{\partial t} + \phi\left(x, t, u\right) \frac{\partial}{\partial u}, \tag{2.4}$$

in which ξ, τ, ϕ are infinitesimal generators. The second-order prolongation for equation (1.1) is

$$Pr^{(2)}v = v + \phi^x \frac{\partial}{\partial u_x} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{tt} \frac{\partial}{\partial u_{tt}}, \tag{2.5}$$

where

$$\begin{aligned} \phi^x &= D_x\phi - u_x D_x \xi - u_t D_x \tau, \\ \phi^{xx} &= D_x^2\left(\phi - \xi u_x - \tau u_t\right), \\ \phi^{tt} &= D_t^2\left(\phi - \xi u_x - \tau u_t\right), \end{aligned} \tag{2.6}$$

where D_x and D_t are fully differentiable operators with respect to x, t.

2.2. Lie symmetry analysis

First, we consider a one-parameter Lie group of point transformation:

$$\begin{aligned} \widetilde{x} &= x + \varepsilon\xi + O\left(\varepsilon^2\right), \\ \widetilde{t} &= t + \varepsilon\tau + O\left(\varepsilon^2\right), \\ \widetilde{u} &= u + \varepsilon\phi + O\left(\varepsilon^2\right), \end{aligned}$$

in which ε is the group parameter, and ξ, τ, ϕ are infinitesimal variables of x, t, u.

Substituting (2.6) into (2.5) and using Maple software to solve the determining equations, the infinitesimal can be deduced as follows.

$$\xi = k_1 x + k_2, \quad \tau = k_3 t + k_4, \quad \phi = \frac{\left(k_1 - k_3\right)\left(1+u\right)}{a}, \tag{2.7}$$

where $k_i (i = 1, \cdots, 4)$ are constants.

Thus, the Lie algebra of equation (1.1) is generated by four generators

$$v_1 = x\frac{\partial}{\partial x} + \frac{1}{a}\left(1+u\right)\frac{\partial}{\partial u}, \quad v_2 = \frac{\partial}{\partial x}, \quad v_3 = t\frac{\partial}{\partial t} - \frac{1}{a}\left(1+u\right)\frac{\partial}{\partial u}, \quad v_4 = \frac{\partial}{\partial t}. \tag{2.8}$$

Substituting (2.8) into (2.4), we obtain

$$v = k_1\left(x\frac{\partial}{\partial x} + \frac{1}{a}\left(1+u\right)\frac{\partial}{\partial u}\right) + k_2\frac{\partial}{\partial x} + k_3\left(t\frac{\partial}{\partial t} + \frac{1}{a}\left(-1-u\right)\frac{\partial}{\partial u}\right) + k_4\frac{\partial}{\partial t}. \tag{2.9}$$

Next, in order to get the one-parameter transformation group of equation (1.1), the system of ODEs with initial values needs to be solved

$$\frac{d}{d\varepsilon}\left(\tilde{x},\tilde{t},\tilde{u}\right)=\psi\left(\tilde{x},\tilde{t},\tilde{u}\right),\ \left(\tilde{x},\tilde{t},\tilde{u}\right)\big|_{\varepsilon=0}=\left(x,t,u\right).$$

The corresponding one-parameter transformation group of the equation is

$$\begin{cases} K_1:\left(e^{\varepsilon}x,t,\left(-1+(1+u)\,e^{\frac{\varepsilon}{a}}\right)\right),\\ K_2:(x+\varepsilon,t,u),\\ K_3:\left(x,e^{\varepsilon}t,\left(-1+(1+u)\,e^{-\frac{\varepsilon}{a}}\right)\right),\\ K_4:(x,t+\varepsilon,u)\,. \end{cases}$$

If $u=f(x,t)$ is the solution of equation (1.1), then the following function is also the solution of equation (1.1)

$$\begin{cases} u^{(1)}=-1+\left(1+f\left(xe^{-\varepsilon},t\right)\right)e^{\frac{\varepsilon}{a}},\\ u^{(2)}=f\left(x-\varepsilon,t\right),\\ u^{(3)}=-1+\left(1+f\left(x,te^{-\varepsilon}\right)\right)e^{-\frac{\varepsilon}{a}},\\ u^{(4)}=f\left(x,t-\varepsilon\right). \end{cases}$$

2.3. Construction of the optimal system

In the following, we construct the optimal system of equation (1.1) using the commutator table and adjoint representation table. According to the definition of Lie bracket and adjoint representation,

$$[v_m,v_n]=v_mv_n-v_nv_m,$$

$$Ad\left(\exp\left(\varepsilon\right)v_m\right)v_n=v_n-\varepsilon\left[v_m,v_n\right]+\frac{1}{2}\varepsilon^2\left[v_m,\left[v_m,v_n\right]\right]-\cdots.$$

We can get the following two tables respectively.

Table 1. Commutator table

$[v_i,v_j]$	v_1	v_2	v_3	v_4
v_1	0	$-v_2$	0	0
v_2	v_2	0	0	0
v_3	0	0	0	$-v_4$
v_4	0	0	v_4	0

Using the adjoint table to give the classification of the subalgebras of the vector fields (2.8), consider the vector

$$V=a_1v_1+a_2v_2+a_3v_3+a_4v_4. \tag{2.10}$$

First, assume that $a_1\neq 0$, and take $a_1=1$,

$$V=v_1+a_2v_2+a_3v_3+a_4v_4.$$

Table 2. Adjoint representation table

Ad	v_1	v_2	v_3	v_4
v_1	v_1	$e^{\varepsilon}v_2$	v_3	v_4
v_2	$v_1-\varepsilon v_2$	v_2	v_3	v_4
v_3	v_1	v_2	v_3	$e^{\varepsilon}v_4$
v_4	v_1	v_2	$v_3-\varepsilon v_4$	v_4

We act $Ad(exp(a_2v_2))$ on V,

$$V^{(1)} = Ad\left(\exp\left(a_2v_2\right)\right)v = v_1 + a_3v_3 + a_4v_4.$$

Then, acting $Ad(exp(\frac{a_4}{a_3})v_4)$ to $V^{(1)}$,

$$V^{(2)} = Ad\left(\exp\left(\frac{a_4}{a_3}v_4\right)\right)V^{(1)} = v_1 + a_3v_3. \tag{2.11}$$

Therefore, if $a_1 \neq 0$, every one-dimensional subalgebra generated by V is equal to $v_1 + a_3v_3$.

Secondly, assuming that $a_1 = 0$, $a_3 \neq 0$, then we assume that $a_3 = 1$. That is,

$$V = a_2v_2 + v_3 + a_4v_4.$$

Then act $Ad(exp(a_4v_4))$ to V,

$$V^{(3)} = Ad\left(\exp\left(a_4v_4\right)\right)V = a_2v_2 + v_3. \tag{2.12}$$

In the third case, when $a_1 = 0, a_3 = 0$, and $V = a_2v_2 + a_4v_4$, there are four cases

$$v_2, v_2 \pm v_4, v_4. \tag{2.13}$$

According to (2.11), (2.12) and (2.13), we obtain that the one-dimensional optimal system of the four subalgebras (2.8) is spanned by: (a) $v_1 + a_3v_3$, (b) $a_2v_2 + v_3$, (c) $v_2 + v_4$, (d) $v_2 - v_4$, (e) v_2, (f) v_4.

2.4. Exact solutions for equation (1.1)

In this section, the symmetry reductions and exact solutions are studied for equation (1.1).

Case 1. For generator

$$V = v_1 + a_3v_3 = x\frac{\partial}{\partial x} + \frac{1}{a}\left(1+u\right)\frac{\partial}{\partial u} + a_3\left(t\frac{\partial}{\partial t} - \frac{1}{a}\left(1+u\right)\frac{\partial}{\partial u}\right),$$

the characteristic equation satisfies

$$\frac{dx}{x} = \frac{dt}{a_3t} = \frac{du}{\frac{1-a_3}{a}\left(1+u\right)}. \tag{2.14}$$

The group invariant solution is

$$u = -1 + t^{\frac{1-a_3}{aa_3}}f\left(h\right), \tag{2.15}$$

in which $h = \dfrac{x^{a_3}}{t}$. Assuming that $a_3 = 1$, $a = 1$ and substituting (2.15) into equation (1.1), we obtain

$$-2f(f')^2 + f''\xi^2 - f^2 f'' + 2f'\xi = 0. \tag{2.16}$$

Integrating equation (2.16),

$$\left(f^2 - \xi^2\right) f' = 0, \tag{2.17}$$

and then we have

$$f = -\xi, f = \xi, f = c_1.$$

Therefore, equation (1.1) has self-similar solutions as $u = -1 - \dfrac{x}{t}$, $u = -1 + \dfrac{x}{t}$ and $u = -1 + k_1$.

Case 2. For generator

$$V = a_2 v_2 + v_3 = a_2 \frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{1}{a}\left(1+u\right)\frac{\partial}{\partial u},$$

the characteristic equation satisfies

$$\frac{dx}{a_2} = \frac{dt}{t} = \frac{du}{-\frac{1}{a}\left(1+u\right)}.$$

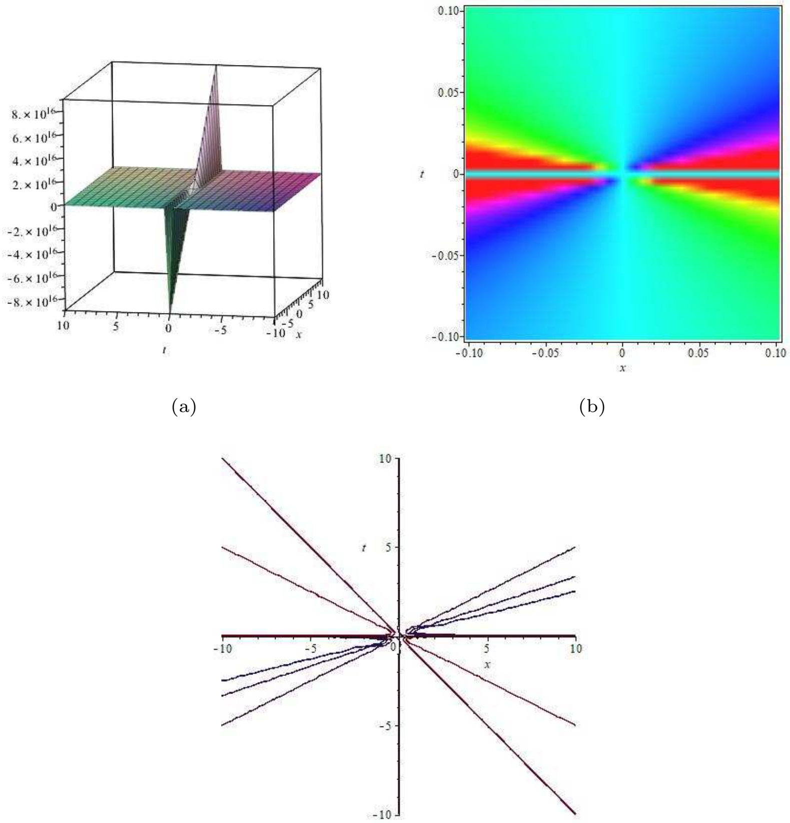
The group invariant solution is

$$u = -1 + t^{-\frac{1}{a}} g\left(\xi\right),$$

where the invariant $\xi = ce^{\frac{x}{b}}$. Taking it into equation (1.1),

$$g^2 g''\xi^2 + 2g(g')^2\xi^2 + g^2 g'\xi - g''\xi^2 - 4g'\xi - 2g = 0. \tag{2.18}$$

Equation (2.18) is a completely nonlinear ordinary differential equation, which is not easy to solve.



(a) (b)

(c)

Figure 1. The dynamical structures of $u = -1 + \frac{x}{t}$: (a) singularity profile of $u = -1 + \frac{x}{t}$; (b) density plot; (c) contour plot

Case 3. When $V = v_2 + v_4 = \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial t}$, the characteristic equation is $\dfrac{dx}{1} = \dfrac{dt}{1}$. The traveling wave solution is $\xi = \xi(x-t)$, where u can be expressed as $u = f(x-t)$. Then, taking it into equation (1.1),

$$f' = (1+f)^{2a} f' - k_1, \tag{2.19}$$

in which a, k_1 are arbitrary constants. Integrating (2.19), we get

$$\frac{(1+f)^{2a+1}}{2a+1} - f = \frac{k_1 \xi + k_2}{2a+1}, \tag{2.20}$$

$$(1+f)^{2a+1} - (2a+1) f = k_1 \xi + k_2, \tag{2.21}$$

where k_1 and k_2 are arbitrary constants.

Case 4. When $V = v_2 = \dfrac{\partial}{\partial x}$, the solution of equation (1.1) can be expressed as $u = f(t)$. For this equation, $f'' = 0$, we can obtain $f = k_1 t + k_2$, where k_1 and k_2 are arbitrary constants.

Case 5. When $V = v_4 = \dfrac{\partial}{\partial t}$, the invariant is $\xi = x$, and then $u = f(x)$. Taking it into equation (1.1), we obtain

$$(1+f)^{2a} f' = k_1, \tag{2.22}$$

in which k_1 is an arbitrary constant. Then, integrating equation (2.22), we get

$$(1+f)^{2a+1} = (2a+1)\cdot k_1 x.$$

3. Analysis of conservation laws of equation (1.1)

In this section, we discuss the conservation laws of this one-dimensional nonlinear wave equation. In [17], some definitions of conservation laws are given.

Definition 3.1 Let x, t be the independent variables, $u = u(x,t)$ be the dependent variable, and u_x, u_t , $u_x x$, $u_t t$, etc, be its partial derivative. Next, we introduce the conjugate equation and the definition of the multiplier method.

Assume that the s-order partial differential equation with m independent variable can be expressed as

$$F\left(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(s)}\right) = 0, \tag{3.1}$$

where $x = (x_1, x_2, \cdots, x_m)$, $u_{(i)}$ represents be derivatives of u with respect to $x_1, x_2, \cdots, x_n$.

The conjugate equation of equation (3.1) is $F^* = (x, t, u, v, u_x, v_x, u_t, v_t, u_{xx}, \cdots)$,

$$F^*\left(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, \cdots, u_{(s)}, v_{(s)}\right) = \frac{\delta\left(\Lambda F\right)}{\delta u}. \tag{3.2}$$

The operator $\dfrac{\delta}{\delta u} = \dfrac{\partial}{\partial u} - D_i \dfrac{\partial}{\partial u_i} + D_{ij}\dfrac{\partial}{\partial u_{ij}} - D_{ijk}\dfrac{\partial}{\partial u_{ijk}} + D_{ijkl}\dfrac{\partial}{\partial u_{ijkl}} - \cdots$ is the Euler-Lagrange operator, and D_i represents the total differentiation of x_i.

Definition 3.2 Equation (3.1) and equation (3.2) have a Lagrange

$$\mathscr{L} = F = \Lambda\left(F\left(x, u, u_{(1)}, u_{(2)}, u_{(3)} \cdots, u_{(s)}\right)\right), \tag{3.3}$$

where $\mathscr{L}$ meets $\frac{\delta \mathscr{L}}{\delta u} = F^*$, $\frac{\delta \mathscr{L}}{\delta v} = F$. For equation (3.1), its conservation laws can be expressed as

$$\begin{aligned} M^i =& \xi^i \mathscr{L} + W^\alpha [\frac{\partial \mathscr{L}}{\partial u_i^\alpha} - D_j\left(\frac{\partial \mathscr{L}}{\partial u_{ij}^\alpha}\right) + D_j D_k\left(\frac{\partial \mathscr{L}}{\partial u_{ijk}^\alpha}\right) - D_j D_k D_m\left(\frac{\partial \mathscr{L}}{\partial u_{ijkm}^\alpha}\right) \\ & + \cdots] + D_j\left(W^\alpha\right)\left[\frac{\partial \mathscr{L}}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial \mathscr{L}}{\partial u_{ijk}^\alpha}\right) + D_k D_m\left(\frac{\partial \mathscr{L}}{\partial u_{ijkm}^\alpha}\right) - \cdots\right] \\ & + D_j D_k\left(W^\alpha\right)\left[\frac{\partial \mathscr{L}}{\partial u_{ijk}^\alpha} - D_m\left(\frac{\partial \mathscr{L}}{\partial u_{ijkm}^\alpha}\right) + \cdots\right] \\ & + D_j D_k D_m\left(W^\alpha\right)\left[\frac{\partial \mathscr{L}}{\partial u_{ijkm}^\alpha} - D_n\left(\frac{\partial \mathscr{L}}{\partial u_{ijkmn}^\alpha}\right) + \cdots\right], \end{aligned} \tag{3.4}$$

in which $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$.

The formal Lagrangian of equation (1.1) is

$$\mathscr{L} = \Lambda\left(x, t, u, u_x, u_t\right)\left(u_{tt} - \left((1+u)^{2a} u_x\right)_x\right). \tag{3.5}$$

Assuming that $\Lambda = k_1 x + k_2 t + k_3 xt$, we obtain the following four cases.

Case 1. For generator $v_1 = x\dfrac{\partial}{\partial x} + \dfrac{1}{a}\left(1+u\right)\dfrac{\partial}{\partial u}$, the Lie characteristic function is $W = \dfrac{1}{a}\left(1+u\right) - xu_x$,

$$M^x = \left(\frac{1+u}{a} - xu_x\right)\left(-4avu_x(1+u)^{2a-1} + v_x(1+u)^{2a} + \frac{2avu_x(1+u)^{2a}}{1+u}\right)$$
$$- v(1+u)^{2a}\left(\frac{u_x}{a} - xu_{xx} - u_x\right),$$
$$M^t = -v_t\left(\frac{1+u}{a} - xu_x\right) + v\left(\frac{u_t}{a} - xu_{xt}\right).$$

Case 2. For generator $v_2 = \dfrac{\partial}{\partial x}$, the Lie characteristic function is $W = -u_x$,

$$M^x = -u_x\left(-\frac{2avu_x(1+u)^{2a}}{1+u} + v_x(1+u)^{2a}\right) + u_{xx}v(1+u)^{2a},$$
$$M^t = u_x v_t - vu_{xt}.$$

Case 3. For generator $v_3 = t\dfrac{\partial}{\partial t} + \dfrac{1}{a}\left(-1-u\right)\dfrac{\partial}{\partial u}$, the Lie characteristic function is $W = -\dfrac{1}{a}\left(1+u\right) - tu_t$,

$$M^x = \left(-\frac{1+u}{a} - tu_t\right)\left(-\frac{2av(1+u)^{2a}u_x}{1+u} + v_x(1+u)^{2a}\right)$$
$$- v(1+u)^{2a}\left(-\frac{u_x}{a} - tu_{xt}\right),$$
$$M^t = -v_t\left(-\frac{1+u}{a} - tu_t\right) + v\left(-\frac{u_t}{a} - tu_{tt} - u_t\right).$$

Case 4. For generator $v_4 = \dfrac{\partial}{\partial t}$, the Lie characteristic function is $W = -u_t$,

$$M^x = \frac{2avu_x u_t(1+u)^{2a}}{1+u} + vu_{xt}(1+u)^{2a},$$
$$M^t = -vu_{tt}.$$

4. Conclusions

We analyze the symmetry of one-dimensional nonlinear wave equation by classical Lie symmetry method, and then the optimal system of the symmetry are derived. By solving the reduced equation, we can calculate the solutions of the equation. Finally, the conservation laws have been established through the use of conservation law multiplier.

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