Journal of Nonlinear Modeling and Analysis http://jnma-online.com
Volume 6, Number 2, June 2024, 305–319 DOI:10.12150/jnma.2024.305

Optimal Decay Rates for the Highest-Order Derivatives of Solutions for the Compressible MHD Equations with Coulomb Force*

Liuna Qin and Zhengyan Luo†

Abstract For the Cauchy problem of the 3D compressible MHD equations with Coulomb force, the large time behavior of this model is further investigated in this article. Compared to the previous related works in Tan-Tong-Wang [*J. Math. Anal. Appl.* 427 (2015) 600–617], the main novelty of this paper is that we prove the optimal decay rates for the highest-order spatial derivatives of the solutions to the compressible MHD equations with Coulomb force, which are the same as those of the heat equation.

Keywords MHD equations, highest-order derivatives, optimal decay rates

MSC(2010) 35Q35, 35B40, 76W05.

1. Introduction

Considering the initial value problem of 3D isentropic MHD equations with Coulomb force for viscous compressible fluid:

$$\begin{cases} \rho_t + \mathrm{div}(\rho v) = 0, \\ (\rho v)_t + \mathrm{div}(\rho v \otimes v) + \nabla P = curl H \times H + \mu \Delta v + (\lambda + \mu)\nabla \mathrm{div} v + \rho \nabla \Phi, \\ H_t - curl(v \times H) - \nu \Delta H = 0, \quad \mathrm{div} H = 0, \\ \Delta \Phi = \rho - \bar{\rho}, \end{cases} \tag{1.1}$$

with the initial data as follows:

$$(\rho, v, H, \nabla\Phi)(t,x)\big|_{t=0} = (\rho_0, v_0, H_0, \nabla\Phi_0)(x), \tag{1.2}$$

the far field behavior of solutions, we assume:

$$(\rho_0, v_0, H_0, \nabla\Phi_0)(x) \to (\bar{\rho}, 0, 0, 0) \quad \text{as } |x| \to \infty. \tag{1.3}$$

Here the unknown functions $\rho = \rho(t,x) \geq 0$, $v = v(t,x) \in \mathbb{R}^3$, $H = H(t,x) \in \mathbb{R}^3$ and $\Phi = \Phi(t,x) \in \mathbb{R}^3$ are density, velocity, magnetic field and electric potential

†the corresponding author.
Email address:liunaqin321@stu.gxnu.edu.cn.(L.Qin),zhengyanluo33@163.com (Z. Luo)
School of Mathematics and Statistics, Guangxi Normal University, Guilin, Guangxi 541004, P.R. China
*The authors were supported by National Natural Science Foundation of China (No. 12271114) , Guangxi Natural Science Foundation (No. 2019JJG110003, No. 2019AC20214) and Innovation Project of Guangxi Graduate Education (No. YCSW2022163).

respectively. The pressure $P = P(\rho)$ is a smooth function with $P^{'}(\rho) > 0$ for $\rho > 0$. The constants μ, λ are the viscosity coefficients of the flow, and they satisfy the physical restrictions $\mu > 0$ and $2\mu + 3\lambda \geq 0$. The constant $\nu > 0$ represents the magnetic diffusivity. $\bar{\rho}$ represents the positive constant.

1.1. History of the problem

Let us give some explanations about the above model. When the magnetic field and the Coulomb force are taken into account, the compressible Navier-Stokes-Poisson equations are transformed into the compressible viscous magnetohydrodynamic(MHD) equations with Coulomb force, which is of hyperbolic-parabolic-elliptic mixed type. Owing to the physical importance and mathematical challenges, there is a huge literature on the investigations of well-posedness of smooth or weak solutions to the compressible MHD equations with Coulomb force, cf. [4,6,8,10,12]. To get straight to the point of this article, let's just give a brief overview of the results for this model. When time goes to infinity, Tan-Wang [8] showed the vanishing vacuum phenomena of the finite energy weak solutions via the weak convergence method. Wang [9] proved that the weak solutions decay exponentially to the equilibrium state in L^2 norm. What'more, Zheng-Tan [10] obtained the optimal time decay estimate of the solutions by spectral analysis and energy methods. And Tan-Tong-Wang [11] obtained the global existence and time decay rates of the solutions through a general energy method. More precisely, under the assumptions that the initial data $\|(\rho_0-\bar{\rho}, v_0, H_0, \nabla\Phi_0)(x)\|_{H^3}$ is sufficiently small, one has

$$\begin{aligned}&\|(\rho-\bar{\rho}, v, H, \nabla\Phi)(t)\|_{H^N}^2 + \int_0^t \|(\rho-\bar{\rho}, \nabla v, \nabla H, \nabla\nabla\Phi)(s)\|_{H^N}^2 \mathrm{d}s \\ \leq & C\|(\rho_0-\bar{\rho}, v_0, H_0, \nabla\Phi_0)\|_{H^N}^2.\end{aligned} \tag{1.4}$$

Furthermore, if $(\rho_0-\bar{\rho}, v_0, H_0, \nabla\Phi_0)(x) \in H^N \cap L^1$, it holds that

$$\|\nabla^\ell(\rho-\bar{\rho}, v, H, \nabla\Phi)(t)\|_{H^{N-l}} \lesssim (1+t)^{-\frac{3}{4}-\frac{\ell}{2}} \quad \ell = 0, 1, ..., N-1, \tag{1.5}$$

and

$$\|\nabla^\ell(\rho-\bar{\rho})(t)\|_{L^2} \lesssim (1+t)^{-(\frac{5}{4}+\frac{l}{2})} \quad \ell = 0, 1, ..., N-2. \tag{1.6}$$

However, when taking $\ell = N-1$ in (1.5), we find that the L^2–decay rate of the highest-order (i.e. N–order) spatial derivative of the solution $(\rho-\bar{\rho}, v, H, \nabla\Phi)$ is the same as that of its lower order, which is $(1+t)^{-\frac{3}{4}-\frac{N-1}{2}}$ and is slower than L^2-rate $(1+t)^{-\frac{3}{4}-\frac{N}{2}}$. Comparing the related results with those of the heat equation, it does not appear to be optimal. Therefore, to improve the time decay rate in (1.5) is an interesting and meaningful problem.

1.2. Notation

Throughout this paper, we denote $L^p(\mathbb{R}^3)$ and $H^k(\mathbb{R}^3)$ as usual Lebesgue space and Sobolev spaces with norm $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^k}$ respectively. In order to construct the low–high frequency decomposition, we will introduce some symbols in the frequency space. Let $\phi_j \in C_0^\infty(\mathbb{R}^3_\xi)(j = 1, \infty)$ be the following cut–off functions:

$$\phi_1(\xi) = \begin{cases} 1 & |\xi| \leq a_0, \\ 0 & |\xi| \geq A_0, \end{cases} \tag{1.7}$$