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# Threshold of Effective Degree SIR Model*

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**Abstract** The effective degree SIR model is a precise model for the SIR disease dynamics on a network. The original ODE model is only applicable for a network with finite degree distributions. The new generating function approach rewrites with model as a PDE and allows infinite degree distributions. In this paper, we first prove the existence of a global solution. Then we analyze the linear and nonlinear stability of the disease-free steady state of the PDE effective degree model, and show that the basic reproduction number still determines both the linear and the nonlinear stability. Our method also provides a new tool to study the effective degree SIS model, whose basic reproduction number has been elusive so far.

**Keywords** Generating function, effective degree model, basic reproduction number, spectral stability, nonlinear stability, steady states

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## 1. Introduction

Classical compartmental models (see, e.g., [1, 7]) assume random mixing, i.e., each pair of individuals has the same rate of contact. Network disease models [8] use contact networks to represent a population and its contacts. Specifically, individuals are represented by nodes and contacts are represented by edges. Such a contact network can model realistic contact patterns in the population such as households and workplaces. It can also be used to study the effectiveness of disease control strategies such as contact tracing and prioritized vaccination.

Early network models are node based, which group individuals by their degrees (the number of contacts) in addition to their infection status (e.g., susceptible, infectious, recovered, etc.) The Sattaros and Vespignani model [11] used such a model to show that, on scale-free networks with an infinite variance of the degree distribution, there exists no disease threshold, i.e., any positive transmission rate

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may cause an outbreak in an infinite population. However, these simple node based models ignore the correlation between the infection status of neighboring nodes, leading to an over-estimate of the disease spread. Lindquist et al. [9] extended these models by grouping the infection status of the nodes and their neighbors, resulting in more precise effective-degree models. Based on effective-degree models, they showed that an SIS type disease (without acquired immunity) is easier to establish than an SIR type model (with acquired immunity), which contrasts the predictions of classical random mixing models that these two types of diseases have identical disease thresholds.

These node based models are systems of ordinary differential equations (ODE) with dimensions dependent on the degree distribution of the network. Model theoretical studies rely on Poisson networks or scale-free networks that have infinite numbers of degrees, resulting in an infinite system. It is a challenge to apply ODE theories to these models. In an earlier paper [6], we have developed a generating function approach to rewrite the effective-degree SIR model to a first-order nonlinear partial differential equation (PDE). We have shown the well-posedness of the PDE model, and shown that, if initially the infection status of the neighbors of a random susceptible node are independent, then the effective-degree PDE model can be simplified to the Volz model [12].

Interestingly, it has been shown (see, e.g., [8]) that, with the same condition on initial conditions, the full SIR model of the pair approximation approach [4, 5] can also be simplified to the Volz model, which can further be simplified to the Miller [10] model. However, if this assumption does not hold, then the Volz-Miller models may not be precise, yet the effective degree models and the pair approximation models may still be applicable. Thus, it is still important to fully understand the effective-degree models.

In this paper, we study the linear and nonlinear stability of the disease-free steady states of the PDE effective-degree SIR model, and show that the disease threshold condition (i.e., the basic reproduction number being unity) that is derived in the finite dimensional effective-degree ODE model is also true for the infinite dimensional model. This disease threshold condition also agrees with that derived from other network models (such as the Volz-Miller models [10, 12].)

In Section 3 we introduce the notation. Section 4 states the main theorems that this paper sets out to prove. Sections 6, 7, and 8 prove the linear stability, nonlinear instability, and nonlinear stability respectively. We provide some discussion and remarks in Section 9.

## 2. The Effective-Degree PDE model

The Effective-Degree ODE model [9] considers an SIR model on a contact network with degree distribution given by $p_k$ where $k$ is the degree. The susceptibles are infected by neighbouring infectious individuals with a per link transmission rate $\beta \geq 0$, and infectious individuals recover to full immunity at a rate $\gamma > 0$. This model compartmentalizes nodes by both their state and by the number of neighbours that it has in each state.

Denote $S_{si}$ as the fraction of susceptible nodes having $s$ susceptible neighbours and $i$ infected neighbours, where $s+i$ is the effective degree of the node, and denote $I$ and $R$ as the fractions of infected and recovered nodes in the population. You may have noticed two things: 1) we don't keep track of the number of recovered