Journal of Nonlinear Modeling and Analysis http://jnma-online.com
Volume 6, Number 4, December 2024, 919–929 DOI:10.12150/jnma.2024.919

Exact Solutions and Optimal System of Hyperbolic Monge-Ampère Equation

Hui Xu [1,†], Minyuan Liu[1]

Abstract Based on Lie symmetry theory, the exact solutions of the hyperbolic Monge-Ampère equation are studied. Firstly, the invariance of the Lie symmetry group is applied to obtain the six-dimensional Lie algebras, then the commutator table and the adjoint representation of the equation are obtained, based on which the optimal system is found. Finally, the exact solutions are obtained by symmetry reduction which transforms the PDEs into easily solvable ODEs.

Keywords Optimal system, exact solution, Lie symmetry, hyperbolic Monge-Ampère equation

MSC(2010) 35L70, 35A09, 35B06.

1. Introduction

Many important scientific and engineering problems can be summarised by the study of nonlinear partial differential equations. Mathematical models in many areas of real life can be described by NLPDEs, and many important fundamental equations in physics, mechanics, fiber optic communication [1], biologic [2] and other disciplines [3] are NLPDEs. Progress has been made in the study of integrable models. For example, Ma [4–6] has proposed many new integrable equations, including coupled modified Korteweg-de Vries four-component equation, coupled nonlinear Schrodinger equation and AKNS type equation, and has achieved breakthrough research results. As research progresses, methods for solving PDEs become more sophisticated. Such as, linear superposition method [7], Bäcklund transformation method [8–11], Darboux transformation method [12–14], Hirota bilinear method [15–21], Inverse scattering method [22–24], Jacobi elliptic function expansion method [25–28], Lie symmetry analysis [29–35], maximum likelihood estimation [36], etc [37–39].

The Monge-Ampère equation was originally formulated by mathematicians Gaspard Monge and André-Marie Ampère in the late eighteenth century and first introduced as a concept in differential geometry. The Monge-Ampère equation is one of the fundamental equations widely used in the fields of elementary unitary calculus and geometric growth. It has demonstrated its importance in this field, not only for solving complex non-linear equations but also for calculating a large number of parameters in fluid mechanics and detecting characteristic parameters

[†]the corresponding author.
Email address:xuhui1232022@163.com(Hui Xu.)
[1]School of Mathematical Sciences, Liaocheng University, Liaocheng 252059, PR China

of electromagnetic and acoustic fields. These properties have made the Monge-Ampère equation very popular in the physical and mathematical community, and its theoretical study and applications have been widely explored [40, 41]. The hyperbolic Monge-Ampère equation is used to describe a class of mechanical systems with singular terminal constraints, and its greatest advantage lies in the fact that its definition can be expressed exactly. Because of the practicality of the hyperbolic Monge-Ampère equation, scholars have done a lot of research. In [42], Kong et al. found that the initial value problem of a one-dimensional hyperbolic mean curvature flow F of a closed plane curve which can be simplified to the initial value problem of a PDE satisfied by the support function of the curve, which is hyperbolic Monge-Ampère equation

$$S_{\tau\tau} = \frac{S_{\theta\tau}^2 - 1}{S + S_{\theta\theta}}.$$

In [43], Michal et al. solved the equivalence problem based on the construction of the hyperbolic Monge-Ampère equation, which had a fundamental differential invariants of the exposure transformation. In [44], Chen proved degenerate hyperbolic Monge-Ampère equation in the existence of smooth solution near zero. Moreover, the linearized equation was transformed into a simpler form by a transformation of variables, leading to its a priori estimate. Finally the existence of local solutions was proved by an iterative method. It was verified that the zero set of small perturbations has a simple structure. What has been studied by the above scholars, as well as the exact solutions in our paper helps us to better understand the structure and properties of the equation.

Nowadays, it is widely used in differential geometry, variational methods, optimization problems, and transmission problems. Its mathematical expression is as follows: the unknown function $z = z(x,t)$ defined in $(x,t) \in R^2$ corresponds to Monge-Ampère equation

$$F + Gz_{xx} + Hz_{xt} + Iz_{tt} + J(z_{xx}z_{tt} - z_{xt}^2) = 0, \tag{1.1}$$

where F, G, H, I, J are first-order variables and x, t, z_x, z_t are the only non-independent variables. F, G, H, I, J depend on x, t, z, z_t, z_x. If conditions

$$\Delta^2(x,t,z,z_t,z_x) \triangleq H^2 - 4GI + 4FJ > 0,$$

and

$$z_{tt} + G(x,t,z,z_t,z_x) \neq 0$$

are satisfied, then (1.1) is hyperbolic.

According to [45], we have

$$\det(D^2 z) = k(x,t)(1 + z_x^2 + z_t^2)^2, \tag{1.2}$$

where $G, H, I = 0$, $F = (1 + z_t^2 + z_x^2)^2$ and $k(x,t)$ stands for the Gaussian curvature of the surface. Hyperbolic Monge-Ampère equations are closely related to geometric applications. A surface with negative Gaussian curvature at each point is a solution to Equation (1.2). We call that a minimal surface, such as a Costa surface. Minimal surfaces cover a wide domain and have negative Gaussian curvature at every point, so the existence of minimal surfaces is closely related to the solvability of the hyperbolic Monge-Ampère equation. Therefore, it is extremely important to find the solutions of the hyperbolic Monge-Ampère equations.