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Asymptotic Expression of Eigenvalues for a Class of Fractional Boundary Value Problems*

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Abstract Based on the properties of Mittag-Leffler function, the behavior of eigenvalue and the eigenfunction for a class of fractional differential equations with integral boundary value condition is discussed, and the asymptotic expression of the eigenfunction is given.

Keywords Fractional differential equation, boundary value problem, eigenvalue, eigenfunction, asymptotic estimation

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1. Introduction

In recent years, fractional differential equations have become a hot topic, as they can describe problems in optical and thermal systems, rheology and material and mechanical systems, signal processing and system identification, control and robotics, and other applications. Many researchers focus on the existence of solutions for fractional differential equations boundary value problems ([1,2], [4,5], [7–10]). For example, in [5], the author considered the existence of solutions for fractional differential equations with integral boundary conditions. However, the theoretical study of fractional differential equations is very difficult because of the nonlocal and singularity of fractional differential operators.

$$\begin{cases} {}^cD^{\alpha}_{0+}y(t) + f(t, y(t)) = 0, t \in (0, 1), \alpha \in (2, 3), & (1.1) \\ y(0) = y'(0) = 0, y(1) = \lambda \int_0^1 y(s)ds, & (1.2) \end{cases}$$

where λ is a parameter and ${}^cD^{\alpha}_{0+}y(t)$ is the standard Caputo fractional derivative. Using the Guo-Krasnoselskii fixed point theorem, the author obtained the sufficient conditions on the existence of positive solutions for problem (1.1)-(1.2). On the other hand, some researchers pay attention to the eigenvalue problem of fractional

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differential equations, see for example [2–4], [6], [12]. In [4], the author mainly studied the eigenvalue problems of fractional order differential equations with Dirichlet and Neumann equal boundary conditions.

$$\begin{cases} {}^cD^{\delta}_{0+}u(t) + \lambda u(t) = 0, t \in (0, a), 1 < \delta < 2, & (1.3)\\ u(0) = u(a) = 0. & (1.4)\end{cases}$$

According to the properties of Caputo fractional derivative and Riemann-Liouville fractional integral, using the Laplace transform, the author shows that the eigenvalue of the fractional order differential equations $\lambda > 0$ is the zero of $f(\lambda) = E_{\delta,2}(-\lambda a^{\delta})$ and the eigenfunction is $u(t, \lambda) = tE_{\delta,2}(-\lambda t^{\delta})$. However, it is difficult to obtain the exact zeros of $f(\lambda)$ and their specific distribution, so the further information of the eigenvalues is not clear, and the specific asymptotic expressions about eigenvalues and eigenfunctions become an important problem to be solved, which inspire us to do this work.

Motivated by [4, 5], using the properties of Mittag-Leffler functions ([10, 11]), we give the specific asymptotic expressions of eigenvalues and eigenfunctions of the following problems (1.5)-(1.6).

$$\begin{cases} {}^cD^{\delta}_{0+}u(t) + \lambda u(t) = 0, t \in (0, 1), 1 < \delta < 2, & (1.5)\\ u(0) = 0, u(1) = \displaystyle\int_0^1 u(s)ds, & (1.6)\end{cases}$$

where ${}^cD^{\delta}_{0+}u(t)$ is a Caputo fractional derivative. As in [4], for a given λ, we call that the nonzero solution $u(t)$ of problems (1.5)-(1.6) is the eigenfunction corresponding to λ, and λ is the eigenvalue.

In order to facilitate the readers, we give the following definitions and lemmas.

Definition 1.1 (Definition 1.1.1, [4]). The Riemann-Liouville fractional integral of $\alpha > 0$ of function $y : (0, \infty) \to R$ is defined as

$$J^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds.$$

Definition 1.2 (Definition 1.1.3, [4]). The Caputo fractional derivative of $\alpha > 0$ of function $f : (0, \infty) \to R$ is defined as

$${}^cD^{\alpha}_{0+}f(t) := J^{n-\alpha}_{0+}D^n(f(t)) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds.$$

Definition 1.3 ([4], [11]). The Mittag-Leffer type function with two parameters is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} (\alpha, \beta > 0).$$

Lemma 1.1 (Lemma 1.1.3, [4]). *Semigroup relations of fractional calculus hold* $J^{\alpha}_{0+}J^{\beta}_{0+}\varphi(t) = J^{\alpha+\beta}_{0+}\varphi(t)$ *in the following circumstances:*

(i) $\beta \geq 0$, $\alpha + \beta \geq 0$, $\varphi(t) \in L^1(0, 1)$;

(ii) $\beta \leq 0$, $\alpha \geq 0$, $\varphi(t) \in J^{-\beta}_{0^+}(L^1(0, 1))$;

(iii) $\alpha \leq 0$, $\alpha + \beta \leq 0$, $\varphi(t) \in J^{-\alpha-\beta}_{0^+}(L^1(0, 1))$.