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On the Cauchy Problem for a Viscous Cahn-Hilliard-Oono System with Chemotaxis

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Abstract In this paper, we are concerned with the well-posedness and large time behavior of Cauchy problem for viscous Cahn-Hilliard-Oono system with chemotaxis in 3D whole space. By using the pure energy method, standard continuity arguments together with negative Sobolev norm estimates, one proves the global well-posedness and time decay estimates.

Keywords Viscous Cahn-Hilliard-Oono system, global existence, decay

MSC(2010) 35K55, 35D35.

1. Introduction

In this paper, we consider the following system of partial differential equations [11] in 3D whole space

$$\partial_t\varphi = \Delta\left(\Psi'(\varphi) - \Delta\varphi - \chi\sigma + \partial_t\varphi\right) - \alpha\varphi, \quad \text{in } \boldsymbol{R}^3 \times (0, +\infty), \tag{1.1}$$

$$\partial_t\sigma = \Delta(\sigma - \chi\varphi), \quad \text{in } \boldsymbol{R}^3 \times (0, +\infty), \tag{1.2}$$

together with the initial conditions

$$\varphi|_{t=0} = \varphi_0,\, \sigma|_{t=0} = \sigma_0, \quad \text{in } \boldsymbol{R}^3. \tag{1.3}$$

Systems (1.1)-(1.2) can be seen as a simplified, fluid-free version of the general thermodynamically consistent diffuse interface model derived in [12]. This model is suitable for a two-phase incompressible fluid mixture with chemical species subject, which are influenced by important mechanisms such as diffusion, chemotactic interactions, and active transport. The order parameter (phase function) φ is the difference in volume fractions between the two components, while the variable σ is the standard for nutrient concentration. $\mu = \Psi'(\varphi) - \Delta\varphi - \chi\sigma + \partial_t\varphi$ is regarded as the chemical potential associated with (φ, σ), in which the function $\Psi'(\varphi)$ is the derivative of a potential Ψ with double-well structure. A physically significant example of Ψ is given by the so-called Flory-Huggins logarithmic potential [8,10,16]

$$\Psi(r) = \frac{\theta}{2}[(1-r)\ln(1-r) + (1+r)\ln(1+r)] + \frac{\theta_0}{2}\left(1 - r^2\right), \forall r \in (-1, 1) \tag{1.4}$$

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with $0 < \theta < \theta_0$. It is referred to as a singular potential since its derivative $\Psi'(\varphi)$ blows up at the pure phases ± 1. In the literature, the singular potential Ψ is often approximated by a fourth order polynomial [17,18]

$$\Psi(r) = \frac{1}{4}\left(1 - s^2\right)^2, \quad \forall s \in R. \tag{1.5}$$

The nontrivial coupling between Cahn-Hilliard equations (1.1) and the diffusion equation (1.2) for the nutrient is characterized by the constant χ, which models some specific mechanisms such as chemotaxis/active transport in the context of tumor growth modeling (see, e.g., Garcke and Lam [6] and Garcke et al [5]). Cahn-Hilliard equation (1.1) also involves some nonlocal interaction that is given by Oono's type $-\alpha\varphi$ for the sake of simplicity(cf., e.g., Giorgini et al. and Miranville [7,13]), where $\alpha \geqslant 0$. Recently, He [11] considered the properties of solutions for the initial-boundar value problem of equations (1.1)-(1.2) with singular potentials including the physically relevant logarithmic potential in the 3D bounded domain. The authors proved the existence and uniqueness of a global weak solution, obtained some regularity properties of the weak solution when $t > 0$, and studied the longtime behavior of the system. We remark that there is no paper related to the Cauchy problem of equations (1.1)-(1.2). This is just the main purpose of this paper. In this paper, we consider the global existence and long time behavior of global strong solutions for equations (1.1)-(1.2) in 3D whole space.

Remark 1.1. If $\sigma = 0$ in equations (1.1)-(1.2), we obtain the well-known classical Cahn-Hilliard equation, which has been employed as an efficient mathematical tool for the study on dynamics of binary mixtures, particularly, recently for the tumor growth modeling [3, 9, 15]. Concerning the mathematical analysis of the Cahn-Hilliard equation and its variants, we refer to several studies [1, 2, 4, 13]. And the references cited therein (see also the recent book [14]).

Our main result is stated as follows:

Theorem 1.1. *Assume that $\varphi_0, \sigma_0 \in R^3$ for an integer $N \geq 2$ and $2\sqrt{\alpha} - 1 - 2\chi^2 > 0$. Then there exists a constant δ_0 such that if*

$$\|\varphi_0\|_{H^2} + \|\nabla\varphi_0\|_{H^2} + \|\sigma_0\|_{H^2} < \delta_0, \tag{1.6}$$

then problem (1.1) admits a unique global solution (φ, σ) satisfying that for all $t \geq 0$,

$$\begin{aligned} &\|\varphi(t)\|_{H^N}^2 + \|\nabla\varphi(t)\|_{H^N}^2 + \|\sigma(t)\|_{H^N}^2 \\ &+ \int_0^t \left(\|\nabla\varphi(\tau)\|_{H^N}^2 + \left\|\nabla^2\varphi(\tau)\right\|_{H^N}^2 + \|\nabla\sigma(\tau)\|_{H^N}^2\right) d\tau \\ \leq& C\left(\|\varphi_0\|_{H^N}^2 + \|\nabla\varphi_0\|_{H^N}^2 + \|\sigma_0\|_{H^N}^2\right). \end{aligned} \tag{1.7}$$

If further, $\varphi_0, \sigma_0 \in \dot{H}^{-s}$ for some $s \in \left[0, \frac{3}{2}\right)$, then for all $t \geq 0$,

$$\|\varphi(t)\|_{\dot{H}^{-s}}^2 + \|\nabla\varphi(t)\|_{\dot{H}^{-s}}^2 + \|\sigma(t)\|_{\dot{H}^{-s}}^2 \leqslant C_0, \tag{1.8}$$

and the following decay result holds for $l = 0, \cdots, N-1$:

$$\left\|\nabla^l\varphi(t)\right\|_{H^{N-l}} + \left\|\nabla^{l+1}\varphi(t)\right\|_{H^{N-l}} + \left\|\nabla^l\sigma(t)\right\|_{H^{N-l}} \leqslant C_0(1+t)^{-\frac{l+s}{2}}.$$