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Exact Solution to the Compressible Euler System in 1-D*

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Abstract In this paper, the exact solution of one-dimensional isentropic Euler equations is studied. When the exponent of the state equation satisfies $\gamma = 2$, we get an exact solution which is linear with respect to the spatial variable x. For this end, we solve some ordinary differential equations with time dependent variable coefficients.

Keywords Euler equations, compressible, exact solutions, ordinary differential equation

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1. Introduction

As the most basic equations in the field of fluid mechanics, the compressible Euler system can be used to describe and simulate many physical phenomena in real fluids. The long time behavior of the solution to the equations has been widely studied. Although there is no complete explanation so far, the research in this aspect is going on all the time. There have been a lot of results. The isentropic compressible Euler equations in n-d are as follows

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^{\mathbf{n}}, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p = 0, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^{\mathbf{n}}, \end{cases} \tag{1.1}$$

where $\rho = \rho(t,x)$, $u = u(t,x) = (u_1(t,x), u_2(t,x), \cdots, u_n(t,x))$ and $p = p(t,x)$ as the real value of the unknown functions, represent the density, velocity and pressure of the fluid, respectively.

System (1.1) is a nonlinear system of partial differential equations. General speaking, it is very difficult to obtain the exact solution. The exact solution results are mainly for one-dimensional Euler equations or high dimensional radial symmetry cases so far. Li and Wang considered radially symmetric solutions in any dimension and constructed an exact solution of the form $u = c(t)r$ (where $r = |x|$ and $c(t)$ satisfies a second-order ordinary differential equation) and the blow up of the solution of the compressible Euler equations was further analyzed [1]. Liang constructed an exact solution to the non-isentropic one-dimensional Euler equations in the form of $u = c(t)x$ [2]. Furthermore, the blow up result is discussed by setting

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appropriate initial values $c(0)$ and $c'(0)$. Using similar methods, Yuen obtained the following analytic solution in one-dimensional case:

$$\begin{cases} \rho^{\gamma-1}(x,t) = \max\left\{\rho^{\gamma-1}(0,t) - \frac{\gamma-1}{K\gamma}\left[\dot{b}(t) + b(t)\frac{\dot{a}(t)}{a(t)}\right]x \right. \\ \left. -\frac{(\gamma-1)\xi}{2K\gamma a^{\gamma+1}(t)}x^2, 0\right\}, \\ u(x,t) = \frac{\dot{a}(t)}{a(t)}x + b(t), \end{cases} \tag{1.2}$$

where $a(t)$, $b(t)$, and $c(t)$ satisfy a system of differential equations [3]. Dong and Li obtained radially symmetric and self-similar analytic solutions to compressible Euler equations with time dependent damping and free boundary in three dimensional space, and proved the global existence of such solutions [4]. Jia considered one-dimensional isothermal Euler equations ($p = a\rho$) with time dependent damping, and an exact solution of the form $\rho\,(x,t) = e^{c(t)x+d(t)}$ was obtained [5]. For compressible Navier-Stokes equations, there has also been a lot of work on solving the analytical solutions and further studying the large time behavior of the solutions, see [6–8] and the references therein.

The existing results show that the exact analytical solution is possible to obtain in the case of one-dimensional or high-dimensional radial symmetry. In other cases, although it is not easy to obtain the exact solution directly, it is very important to study the large time behavior of the solution. Since the compressible Euler equations can be written in the form of the symmetric hyperbolic system, which admits a common phenomenon of singularity formation. We refer to [9–20] and references therein for such kinds of results.

In this paper, the compressible Euler equations in one-dimension are mainly considered. If the state equation is $p = \frac{1}{2}\rho^2$, by solving an ordinary differential equation, we obtain an exact solution, in which the density $\rho(t,x)$ and the velocity $u(t,x)$ are linear with respect to the space variable x.

Specifically, we consider the following theorem.

$$\begin{cases} \rho_t + (\rho u)_x = 0, (x,t) \in \mathbb{R}\times\mathbb{R}^+, \\ (\rho u)_t + \left(\rho u^2\right)_x + \left(\frac{1}{2}\rho^2\right)_x = 0, (x,t) \in \mathbb{R}\times\mathbb{R}^+. \end{cases} \tag{1.3}$$

Then we have

Theorem 1.1. *For system* (1.3), *there exists an exact solution of the following form*

$$\begin{aligned} &\rho(x,t) = c(t)x + d(t), \\ &u(x,t) = -\frac{c'(t)}{2c(t)}x + \frac{c'(t)d(t) - 2c(t)d'(t)}{2c^2(t)}, \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} &c(t) = \frac{c_2}{(t+2c_1)^2}, \\ &d(t) = \frac{c_3}{t+2c_1} + \frac{c_4 - c_2^2 - c_2^2\log(t+2c_1)}{(t+2c_1)^2}, \end{aligned} \tag{1.5}$$

and c_1, c_2, c_3, c_4 *are constants.*