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Stability of Phase Locking for Bidirectionally Non-symmetric Coupled Kuramoto Oscillators in a Ring*

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Abstract This paper deals with the stability of phase locking for the identical Kuramoto model, where each oscillator is influenced sinusoidally by two neighboring oscillators. By studying the model with bidirectionally non-symmetric coupling in a ring configuration, all phase-locked solutions are comprehensively delineated, and the basin of attraction for the stable phase-locked state is estimated. The stability of these phase-locked solutions is clearly established, highlighting that only the synchronized state and splay-state are stable equilibria. The crucial tools in this work are the standard linearization technique and the nonlinear analysis arguments.

Keywords Coupled oscillators, stability, phase locking, synchronization

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1. Introduction

Background.- The synchronization (in short, sync) process of large populations of weakly coupled oscillators often appears in natural systems and it has received considerable attention because of its application in diverse areas such as biology, neuroscience, engineering, computer science, economy and sociology [1, 2, 5, 12, 19]. Among the many mathematical models of sync, our interest in this paper lies in the Kuramoto model for identical oscillators with a bidirectionally non-symmetric topology. For identical Kuramoto oscillators with all-to-all coupling, a lot of studies have been done for this model, see [3,6,9,17,22]. It is well known that the phase sync is the only stable phase-locked state, which denotes the collapse of all phases into a single phase, see [14]. Hence, almost all initial configurations of phases converge to the phase sync asymptotically. It is reasonable to guess that different asymptotic patterns for Kuramoto oscillators can emerge depending on different network topologies. For example, in [21], Wei et al. consider the periodic sampled-data coupling in the scenario with a generally connected and undirected communication topology, where the connected communication topology means that no isolated oscillator exists in the Kuramoto oscillator network. In [7], Ferguson studies bifurcations in the Kuramoto model on a ring network using a novel vector flow and derives criteria for

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certain bifurcations. As network topologies are varied, different stable phase-locked states will emerge asymptotically, and hence it would be an interesting problem to classify all possible stable asymptotic states and initial configurations converging to a given stable equilibrium. This kind of question comes down to the identification of the basin of attractions in dynamical systems theory. There is some literature addressing this issue for the locally coupled Kuramoto model. The literature [14] studies the stability properties of the Kuramoto model with identical oscillators by linear stability analysis and the authors present a six-node example to point out that a stable non-sync equilibrium arises for oscillators bidirectionally coupled in a ring. In [20], Wiley et al. address the problem of "the size of the sync basin" for the locally coupled Kuramoto model with symmetric forward and backward k-neighbor coupling. They have found that when N is the number of oscillators and $\frac{k}{N}$ is greater than a critical value, then the phase sync is the only stable phase-locked state; as $\frac{k}{N}$ passes below this critical value, other stable phase-locked states are born, which takes the form of the splay-state. In [10, 15], the stability of phase-locking is considered for identical Kuramoto oscillators unidirectionally coupled in a ring. In [23], Zhao et al. consider the Kuramoto model with bidirectionally symmetric coupling network and use Łojasiewicz theory to prove the stability of phase-locked solutions. Dong et al. [4] study the interplay of time-delayed interactions and network structure on the collective behaviors of Kuramoto oscillators. In [11], an adaptive control law for inducing in- and antiphase sync in a pair of relaxation oscillators is proposed. Ito et al. show that the phase dynamics of the oscillators coupled by the control law can be reduced to the dynamics of Kuramoto phase oscillators, and they choose a ring topology to show the time series data of states and controlled thresholds for differential initial conditions.

In this work, we study the dynamical behavior of a finite group of Kuramoto oscillators bidirectionally non-symmetric coupled in a ring by performing nonlinear stability analysis. We consider the oscillators labeled as $1, 2, \ldots, N$, with each i-th oscillator coupled by the $(i+1)$-th and $(i-1)$-th oscillators sinusoidally, in which the first oscillator is coupled by the second and N-th oscillators; the last N-th oscillator is influenced by the 1-th and $(N-1)$-th oscillators. This situation appears in many engineering applications and biological modeling of animal locations [8, 13, 18]. For non-symmetric coupling, the total phase is not conserved quantities which causes considerable mathematical difficulty. Of course this lack of symmetry in the coupling makes the asymptotic dynamics of Kuramoto oscillators richer than that of the mean-field case, because there is room for the emergence of other stable phase-locked states other than sync.

Contributions.- The bidirectionally non-symmetric coupling strength makes it difficult to study the phase-locked states for the Kuramoto oscillators in a ring. The contributions of this paper are threefold. First, we identify the formation of all phase-locked states for system (2.3) (see Theorem 2.1). This will enable us to count and classify the phase-locked states. Second, we prove that the sync and splay-state are the only stable phase-locked state (see Theorem 2.2 and Section 3). Third, we present proper subsets of basins of sync and splay-state using nonlinear analysis arguments asymptotically, which says that a given initial configuration will converge to sync or splay-state (see Theorem 3.1).

Organization of paper.- In Section 2, we present the model system and list the formation of all phase-locked states and their stability properties. In Section 3, based on linearization technique and nonlinear analysis arguments, we strictly