Journal of Nonlinear Modeling and Analysis https://journal.global-sci.org/jnma
Volume 7, Number 1, February 2025, 189–208 DOI:10.12150/jnma.2025.189

Numerical Analysis for Fractional Riccati Differential Equations Based on Finite Difference Method

Bowen Xie[1] and Yijin Gao[2,†]

Abstract The fractional Riccati differential equation has a wide application in various areas, for instance, economics and the description of solar activity. In this paper, we focus on the numerical approach of the fractional Riccati differential equations. Two different types of fractional operators are considered under the Riemann-Liouville and Caputo senses. From the numerical simulations, we observe that the explicit finite difference method is not stable. Instead, we employ the implicit finite difference methods to discretize the complicated systems such that stability can be guaranteed. We also exhibit the total error estimations for our algorithms to ensure good approximations. Compared with the other polynomial numerical methods, we can properly extend the model into a larger domain with a large terminal time, which can be verified by numerical examples. Further, we discuss some complex numerical examples to demonstrate the performance of our methods and indicate that our approaches are applicable and tractable to other fractional Riccati equations.

Keywords Fractional Riccati differential equations, finite difference method, implicit method, numerical examples

MSC(2010) 34A08, 65M06, 65N06.

1. Introduction

Riccati differential equation has been used widely in the stochastic control [13, 14, 16, 50, 53] and physics [2, 48, 54]. There are various types of Riccati differential equations and generally, one may see some quadratic forms as in [3, 49]:

$$\frac{dy(t)}{dt} = A(t)y(t) + B(t)y^2(t) + C(t),$$

where $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$ are smooth functions. Its numerical and analytical solutions have been well studied. In [7], the authors established an analytic solution and a reliable numerical approximation of the Riccati equation by using Adomian's decomposition method. [22] presented a method for the computation of the periodic

†the corresponding author.
Email address: bowenx@alumni.iastate.edu (Bowen Xie), yjgao-math@163.com (Yijin Gao)
[1]Department of Mathematics, College of Engineering and Polymer Science, The University of Akron, Akron, OH 44325-4002, US
[2]School of Economics and Finance, Shanghai International Studies University, Shanghai, 201620, PR China

nonnegative definite stabilizing solution of the periodic Riccati equation. Algebraic Riccati equations have been discussed in [5, 20, 21].

As an extension of such a model, a fractional differential operator is employed since the modified equations significantly improve its application in practice (cf. [29, 32, 42, 52]). We consider the following fractional Riccati differential equation (cf. [6]):

$$D^{\alpha}y(t) = A(t)y^2(t) + B(t)y(t) + C(t), \quad 0 < \alpha \leq 1, \quad 0 \leq t \leq 1, \tag{1.1}$$

with a given initial condition, which generally has no analytical solutions (cf. [38, 39]). It is natural to establish alternative numerical methods to study the solution profile.

Several existing methods have been proposed for solving the fractional Riccati differential equation numerically. [26] proposed a modified variational iteration method based on Adomian polynomials. In [31], the authors used a fractional-order Legendre operational matrix. Homotopy perturbation technique and B-spline operational matrix were discussed in [27, 28]. [18] used the hyperbolic-NILT method to solve the fractional differential equations. A new modified Atangana-Baleanu was proposed in [46, 55]. Additionally, the theoretical results regarding the stability were presented in [10, 11]. Other efficient literature can be found in [1, 15, 24, 25, 35, 40, 42, 56].

However, there are some potential constraints within these existing approaches. One may see that we have the limitation for variable t in the equation (1.1), and the polynomial approximation will blow up with $t > 1$. Our major contribution can be summarized as the following: First, we consider a new discretization based on the finite difference method, which expands the original domain for larger t as an extension:

$$D^{\alpha}y(t) = A(t)y^2(t) + B(t)y(t) + C(t), \quad 0 < \alpha \leq 1, \quad t > 1. \tag{1.2}$$

Second, we establish the error estimation under some mild assumptions, which can be verified by its corresponding numerical results. Moreover, our analysis is based on two different fractional operator definitions: Caputo's fractional definition and Riemann-Liouville's fractional definition. Other fractional definitions may be conducted in a similar fashion, and we omit the details for brevity. Last but not least, we consider an implicit finite difference method instead of an explicit method, which loses the stability property in general. However, from our numerical experiments, we observe that the implicit method guarantees the stability of the system, and the graph fits well. Meanwhile, a quadratic solution is provided to manage the implicit component.

The rest of the paper is organized as follows. In Section 2, we present the basic model with two different definitions of fractional operator: Caputo and Riemann-Liouville. Section 3 provides error estimation based on the implicit finite difference method, and the theoretical results indicate stability. Then, Section 4 exhibits the numerical examples to support our theoretical analysis. We draw the conclusions in the Section 5.