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Numerical Solutions for Fractional Burgers' Equation Based on Laplace Transform*

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Abstract The Burgers' equation has widespread applications across various fields. In this paper, we propose an efficient approach for obtaining the numerical solution to the time-fractional Burgers' equation. We extend the classical Burgers' equation to its fractional form by introducing Caputo derivatives. Using the Cole-Hopf transform, we reformulate the problem into a fractional diffusion equation. The Laplace transform method is then applied to convert the equation into an ordinary differential equation (ODE), which can be solved analytically. However, due to the lack of an inverse Laplace transform for this specific form, numerical approximation methods are then utilised to approximate the true solution. Numerical simulations are provided to demonstrate the stability and accuracy of the proposed method.

Keywords Fractional Burgers' equation, Laplace transform, Caputo derivative, numerical simulations

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1. Introduction

Burgers' equation is a fundamental mathematical model extensively used in various fields, including fluid dynamics, traffic flow, and non-linear wave propagation in physics, chemistry, and engineering. Its importance lies in its ability to capture the interaction between non-linear convection and diffusion processes, making it crucial for understanding complex physical phenomena. However, despite its significance, a general analytical solution for this complex system remains elusive, prompting researchers to investigate various numerical algorithms for effective solutions.

Numerous numerical methods have been employed to solve Burgers' equation, including the Finite-Difference Method (FDM), Method of Lines (MOL), Finite-Element Method (FEM), and spline techniques, as highlighted by Bonkile *et al.* [4]. Among FDMs, a key approach involves transforming Burgers' equation into the heat equation using the Hopf-Cole transformation. For example, Kutluay *et al.* converted Burgers' equation into a heat diffusion equation and applied explicit and exact-explicit finite-difference methods to solve the transformed equations under

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specific boundary conditions [23]. Other studies have also leveraged the Hopf-Cole transformation to derive a linear heat equation, yielding promising results [19, 50]. In two-dimensional cases, Bahadir *et al.* introduced a fully implicit finite-difference scheme, solving the non-linear system using Newton's method [3]. Srivastava *et al.* developed a finite-difference technique for coupled viscous Burgers' equations on a uniform grid [48]. The method of lines, initially proposed by Rothe [39], has proven effective in transforming partial differential equations into ordinary differential equation initial value problems. In the context of FEM and spline approaches, Roul *et al.* employed sextic B-spline basis functions for spatial discretisation, achieving highly accurate results with reduced computational time [40], while Majeed *et al.* utilised an extended cubic B-spline collocation scheme for the time-fractional modified Burgers' equation with Caputo fractional derivatives [32]. Dhawan *et al.* provided a comprehensive review of techniques addressing the challenges posed by the non-linear nature of Burgers' equation [8]. Additionally, Cengizci *et al.* stabilized finite element formulations with shock-capturing techniques [6], and Singh *et al.* [46] and Jiwari *et al.* [18] investigated efficient and hybrid methodologies, respectively.

Fractional-order calculus, which extends differentiation to non-integer orders, has gained significant traction in areas such as signal processing, control systems, and mathematical modelling. This is largely due to the time-memory characteristics inherent in fractional derivatives, enabling more accurate modelling of dynamic systems with memory effects. Notable formulations, including the Grünwald-Letnikov [26, 29], Riemann-Liouville [35], and Caputo [10, 31] formulas, are commonly applied. These fractional derivatives have been successfully integrated into ordinary and partial differential equations (ODEs and PDEs), offering new perspectives for addressing complex problems [38, 47].

Incorporating time-fractional derivatives into Burgers' equation allows for the inclusion of memory effects and non-local interactions, which are often present in real-world scenarios but are overlooked in classical models. Analytical solutions to the fractional Burgers' equation are typically limited to specific values of the fractional order parameter, denoted as α. Consequently, efficient numerical methods have been developed to address this limitation, with various studies demonstrating their effectiveness [22, 51]. Finite difference methods have shown particular promise in tackling both time-fractional and space-fractional PDEs [13, 25]. For instance, Chen *et al.* introduced a Fourier method for solving fractional diffusion equations, demonstrating the stability and convergence of their implicit difference approximation scheme [7]. For testing convergence and stability, Roul *et al.* [41] demonstrated the unconditional stability of a proposed non-standard finite difference scheme for the fractional neutron point kinetic equation and utilised Von-Neumann stability analysis for a numerical method applied to the fractional neutron diffusion equation [42]. Other approaches, including finite element methods [17, 27], wavelet methods [49, 52], variational iteration methods [16, 36], homotopy perturbation methods [34], matrix approaches [15, 28], and emerging machine learning techniques [12, 30], have further advanced the solutions for fractional PDEs.

This paper presents a novel approach to numerically solving the fractional Burgers' equation using the Laplace transform. The Laplace transform enables the derivation of an exact solution by directly applying the transform and analytically solving the corresponding ODE. In cases where the inverse Laplace transform lacks an exact solution, we resort to numerical algorithms for computation. The struc-