Journal of Nonlinear Modeling and Analysis
Volume 7, Number 1, February 2025, 334–382
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.334

A General Korteweg-de Vries-Burgers Equation: Novel Ideas and Novel Results

Linghai Zhang

Abstract We consider the Cauchy problem for a general Korteweg-de Vries-Burgers equation and the Cauchy problem for the corresponding linear equation. We will couple together a few novel ideas, several existing ideas and existing results and use rigorous mathematical analysis to accomplish several very important and very interesting results for these Cauchy problems.

Keywords General Korteweg-de Vries-Burgers equation, global smooth solution, existence and uniqueness, sharp rate decay estimates, exact limits, optimal decay estimates

MSC(2010) 35Q20

1. Introduction

1.1. The mathematical model equations and known related results

Consider the Cauchy problem for the following general Korteweg-de Vries-Burgers equation

$$\frac{\partial}{\partial t}u-\alpha\frac{\partial^2}{\partial x^2}u+\beta\frac{\partial^3}{\partial x^3}u+\gamma\mathcal{H}\frac{\partial^2}{\partial x^2}u+\frac{\partial}{\partial x}\mathcal{N}(u)=f(x,t), \tag{1.1}$$

$$u(x,0)=u_0(x). \tag{1.2}$$

Also, consider the Cauchy problem for the corresponding linear equation

$$\frac{\partial}{\partial t}v-\alpha\frac{\partial^2}{\partial x^2}v+\beta\frac{\partial^3}{\partial x^3}v+\gamma\mathcal{H}\frac{\partial^2}{\partial x^2}v=f(x,t), \tag{1.3}$$

$$v(x,0)=u_0(x). \tag{1.4}$$

In these equations, the positive constant $\alpha>0$ represents the diffusion coefficient, the real constants β and γ represent dispersion coefficients, the function $u_0=u_0(x)$ represents the initial function and the function $f=f(x,t)$ represents the external force. Note that the initial functions in both the nonlinear problem and the linear problem are the same, so are the external forces. The Hilbert operator $\mathcal{H}:L^2(\mathbb{R})\to L^2(\mathbb{R})$ is defined by the principal value of the following singular integral

$$[\mathcal{H}\phi](x)=\frac{1}{\pi}\text{ P. V. }\int_{\mathbb{R}}\frac{\phi(y)}{x-y}\mathrm{d}y, \qquad \phi\in L^2(\mathbb{R}).$$

Email address: liz5@lehigh.edu
Department of Mathematics, Lehigh University. 17 Memorial Drive East, Bethlehem, PA 18015 USA

The Fourier transformation of the Hilbert operator $\mathcal{H}$ is given by

$$\widehat{\mathcal{H}\phi}(\xi) = \mathrm{i}\mathcal{S}(\xi)\widehat{\phi}(\xi),$$

for all $\phi \in L^2(\mathbb{R})$ and for all $\xi \in \mathbb{R}$, where $\mathcal{S} = \mathcal{S}(\xi)$ represents the standard sign function

$$\mathcal{S}(\xi) = -1 \text{ for all } \xi < 0, \qquad \mathcal{S}(0) = 0, \qquad \mathcal{S}(\xi) = +1 \text{ for all } \xi > 0.$$

Note that

$$\int_{\mathbb{R}} \phi(x)\mathcal{H}\phi(x)\mathrm{d}x = 0,$$

for all functions $\phi \in L^2(\mathbb{R})$.

The nonlinear function $\mathcal{N} = \mathcal{N}(u) \in C^\infty(\mathbb{R})$. There exists a positive constant $C > 0$, independent of u, such that

$$|\mathcal{N}(u)| \leq C(|u|^2 + |u|^5),$$

for all $u \in \mathbb{R}$. Suppose that there exists the limit

$$\lim_{u \to 0} \frac{\mathcal{N}(u)}{u^2} = \mathcal{L},$$

where $\mathcal{L} \in \mathbb{R}$ is some real constant.

Here are many examples of the nonlinear function

$$\mathcal{N}(u) = u^2, \quad \mathcal{N}(u) = \sin(u^2), \quad \mathcal{N}(u) = \arctan(u^2), \quad \mathcal{N}(u) = \ln(1 + u^2),$$
$$\mathcal{N}(u) = u^2 + u^3, \quad \mathcal{N}(u) = u^2 + u^3 + u^4, \quad \mathcal{N}(u) = u^2 + u^3 + u^4 + u^5.$$

The model equation reduces to the nonlinear Korteweg-de Vries-Burgers equation

$$\frac{\partial}{\partial t}u + \frac{\partial^3}{\partial x^3}u - \alpha\frac{\partial^2}{\partial x^2}u + \frac{\partial}{\partial x}(u^2) = f(x,t),$$

if the nonlinear function $\mathcal{N}(u) = u^2$ and the dispersion coefficients $(\beta, \gamma) = (1, 0)$; it reduces to the nonlinear Benjamin-Ono-Burgers equation

$$\frac{\partial}{\partial t}u + \mathcal{H}\frac{\partial^2}{\partial x^2}u - \alpha\frac{\partial^2}{\partial x^2}u + \frac{\partial}{\partial x}(u^2) = f(x,t),$$

if the nonlinear function $\mathcal{N}(u) = u^2$ and the dispersion coefficients $(\beta, \gamma) = (0, 1)$; and it reduces to the Burgers equation

$$\frac{\partial}{\partial t}u - \alpha\frac{\partial^2}{\partial x^2}u + \frac{\partial}{\partial x}(u^2) = f(x,t),$$

if the nonlinear function $\mathcal{N}(u) = u^2$ and the dispersion coefficients $(\beta, \gamma) = (0, 0)$.

We allow the parameters $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ to be any real constants to include very general cases.

Here are many very important and very interesting questions.