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Error Bounds for Corrected Euler-Maclaurin Formula in Tempered Fractional Integrals

Fatih Hezenci[1,†] and Hüseyin Budak[2]

Abstract In this paper, an equality is established for tempered fractional integrals. With the help of this equality, we prove several corrected Euler-Maclaurin-type inequalities for the case of differentiable convex functions involving tempered fractional integrals. Moreover, we provide our results by using special cases of obtained theorems.

Keywords Quadrature formulae, corrected Maclaurin's formula, tempered fractional integrals, convex functions

MSC(2010) 26D07, 26D10, 26D15, 65D32.

1. Introduction & preliminaries

Simpson-type inequalities are inequalities that are created from Simpson's following rules:

i. Simpson's quadrature formula (Simpson's 1/3 rule) is formulated as follows:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]. \tag{1.1}$$

ii. Simpson's second formula or Newton-Cotes quadrature formula (Simpson's 3/8 rule (cf. [8])) is formulated as follows:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]. \tag{1.2}$$

iii. There is also the corresponding dual Simpson's 3/8 formula - the Maclaurin rule based on the Maclaurin formula (cf. [8]):

$$\int_a^b f(x)\,dx \approx \frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]. \tag{1.3}$$

Formulae (1.1), (1.2) and (1.3) are satisfied for any function f with continuous 4^{th} derivative on $[a,b]$.

The most popular Newton-Cotes quadrature involving three-point is Simpson-type inequality as follows:

†the corresponding author.
Email address:fatihezenci@gmail.com(F. Hezenci),
hsyn.budak@gmail.com(H. Budak)
[1]Department of Mathematics, Faculty of Science and Arts, Duzce University, Duzce 81620, Türkiye.
[2]Department of Mathematics, Faculty of Science and Arts, Kocaeli University, Kocaeli 41001, Türkiye.

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable function on (a,b), and let $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then, one has the following inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.$$

One of the classical closed type quadrature rules is the Simpson 3/8 rule based on the Simpson 3/8 inequality as follows:

Theorem 1.2 (See [8])**.** *If $f:[a,b]\to\mathbb{R}$ is a four times continuously differentiable function on (a,b) and $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$, then one has the inequality*

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\&\leq\frac{1}{6480}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.\end{aligned}$$

The corresponding dual Simpson's 3/8 formula - the Maclaurin rule based on the Maclaurin inequality is as follows:

Theorem 1.3 (See [8])**.** *Assume that $f:[a,b]\to\mathbb{R}$ is a four times continuously differentiable function on (a,b) and $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then, the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\&\leq\frac{7}{51840}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.\end{aligned}$$

Finally, three-point open formula known as corrected Euler-Maclaurin's inequalities (see [15]) is as follows:

Theorem 1.4 (See [15])**.** *Suppose that $f:[a,b]\to\mathbb{R}$ is a four times continuously differentiable function on (a,b) and $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then, it yields*

$$\begin{aligned}&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\&\leq\frac{2401}{28800}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.\end{aligned}$$

Fractional calculus has been grown interest because of its applications in a wide range of disparate domains of science. Because of the importance of fractional calculus, mathematicians have investigated different fractional integral inequalities. Riemann-Liouville fractional integrals, tempered fractional, conformable fractional integrals, and many types of fractional integrals have been considered with several important types of inequalities. The bounds of new formulas can be established by using not only Hermite-Hadamard and Simpson type inequalities but also Newton and Euler-Maclaurin-type inequalities.