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Common Fixed Point of the Commutative F-Contraction Self-mappings with Uniquely Bounded Sequence*

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Abstract We establish the existence of a common fixed point for mappings that satisfy and extend the F-contraction condition. To support our findings, we present pertinent definitions and properties associated with F-contraction mappings. Additionally, we establish an analogue to the Banach contraction theorem. Our results contribute to the broader understanding of this field by extending and generalizing existing findings in the literature.

Keywords Contraction mapping, fixed point, common fixed point

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1. Introduction

In 1976, Jungck [3] pioneered the proof that if two continuous functions, f and g are defined on a complete metric space, with the additional property of being commuting functions and g being an F-contraction such that the range of g is included in that of f, then f and g must possess a unique common fixed point. It is essential to note that the inclusion condition in Jungck's theorem is sufficient but not necessary for the existence of common fixed points. In this study, we maintain all the aforementioned conditions in Jungck's theorem and replace the inclusion, which is an algebraic condition of Jungck's hypotheses, with another topological condition formulated in the form of a bounded Picard sequence. Thus, we obtain the same result as Jungck's theorem but with different hypotheses. In this case, we assure the existence and uniqueness of the common fixed point under the necessary and sufficient conditions.These results are also a generalization of the results discussed in the article [1].

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2. Preliminaries

Let us revisit certain definitions and established results pertaining to common fixed points [6].

Definition 2.1. [1] Consider two metric spaces, (X, d) equipped with distance d. Let f and g be mappings defined from X into itself. A mapping g is deemed an F-contraction if there exists a real positive constant $0 < k < 1$ satisfying the condition:

$$d(g(x), g(y)) \leq kd(f(x), f(y)), \forall x, y \in X. \tag{2.1}$$

We denote this relationship as $g - k - f$. If f is continuous, we refer to $g - k - f$ as a continuous contraction.

The objective of this work is to provide a generalization of the following theorem by eliminating the convergent condition from the hypothesis and streamlining the assumptions.

Theorem 2.1. *[1] Consider a continuous contraction mapping $g - k - f$ in the complete metric space X in itself. Additionally, assume that the mappings f and g commute with each other and there exists an element $x_0 \in X$ such that the Picard sequence $\{f^n(x_0)\}_{n\geq 0}$ converges to $t_0 \in X$. In this case, the Picard sequence $\{g^n(x_0)\}_{n\geq 0}$ converges to a point $r \in X$. Furthermore, the Picard sequence $\{f^n(r)\}_{n\geq 0}$ is bounded, then the sequence $\{g^n(t)\}_{n\geq 0}$ converges to r, which stands as the unique common fixed point of the mappings f and g.*

Remark 2.1. If we substitute the mapping f with an identity mapping in condition (2.1), we retrieve the classical contraction mapping scenario, leading us to the well-known Banach fixed-point theorem, as discussed in [2].

3. Main results and theorems

The proof of our theorems relies on the establishment of certain definitions, properties, propositions, and lemmas.

In what follows, let x_0 denote an element of a non-empty complete metric space X. For the sake of notation, we introduce $g^0(x_0) = x_0, f^0(x_0)) = x_0$ and inductively $g^{n+1}(x_0) = g(g^n(x_0)), f^{n+1}(x_0) = f(f^n(x_0))$, where $n \in \{1, 2, ...\}$.

The results presented in this subsection are commonly referred to as a variant of Banach's contraction principle.

Theorem 3.1. *Let $g - k - f$ be a continuous contraction mapping on the complete metric space X in itself, such that f and g commute with each other. If there exists an element $x_0 \in X$ such that the sequence $\{f^n(x_0)\}_{n\geq 0}$ is bounded, then the maps f and g possess a unique common fixed point in X.*

To establish the proof of the main theorem, we require the following lemmas in succession.

Lemma 3.1. *Let $g-k-f$ be a contraction mapping on the complete metric space X in itself, such that f and g commute with each other. Then the following inequality holds.*

$$d(g^n(x), g^n(y)) \leq k^n d(f^n(x), f^n(y)), \quad \forall x, y \in X, \forall n \in \mathbb{N}. \tag{3.1}$$