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Dynamics of a Diffusive Model with Spatial Memory and Nonlinear Boundary Condition*

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Abstract In this paper, we investigate the existence and stability of steady-state and periodic solutions for a heterogeneous diffusive model with spatial memory and nonlinear boundary conditions, employing Lyapunov-Schmidt reduction and eigenvalue theory. Our findings reveal that when the interior reaction term is weaker than the boundary reaction term, no Hopf bifurcation occurs regardless of time delay. Conversely, when the interior reaction term is stronger than the boundary reaction term, the presence of Hopf bifurcation is determined by the spatial memory delay.

Keywords Spatial memory, stability, Hopf bifurcation, nonlinear boundary condition

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1. Inroduction

Reaction-diffusion systems play a crucial role in both natural sciences and engineering, utilized in biological population dynamics, chemical reactor design, physical studies of material defects, medical disease modeling, and environmental pollutant diffusion. These models enhance our understanding of complex system behaviors and provide a foundation for improving technologies and devising effective strategies. In recent years, extensive research has been conducted on delayed reaction-diffusion equations, particularly focusing on the existence, uniqueness, monotonicity, stability, and bifurcation of steady-state solutions (for example, [2], [5], [8], [9], [11]). Reaction-diffusion models with spatial memory, maturation time, and linear boundary conditions have been extensively explored by Ji and Wu [20], Wang, Fan and Wang [26] and so on. However, research on models containing nonlinear boundary conditions remains limited. Our research includes the impact of external factors such as environmental conditions and resource distribution, which change the dynamical behavior of populations. These factors are crucial in designing more realistic and applicable models, thereby reflecting the complexity of ecological systems. The introduction of nonlinear boundary conditions reflects the complexity of real-world boundary interactions, allowing for a more accurate depiction of system boundary effects and revealing their impact on stability and

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bifurcation phenomena. In this study, we investigate how memory and maturation delays influence the dynamical behavior of nonlinear boundary problems. For convenience, we explore the following system with the memory delay equal to the mature delay:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + d\nabla\cdot\left(u\nabla u_\tau\right) + \lambda u\left(m(x) - u_\tau\right), & x \in \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda h(x,u), & x \in \partial\Omega, \end{cases} \tag{1.1}$$

for $t > 0$, where $\tau > 0$, Δ is the Laplace operator, Ω is a connected bounded open domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, $\mathbf{n}$ is the unit outer normal to $\partial\Omega$, $u = u(x,t) \in \mathbb{R}$, $h \in C^{1+\epsilon}(\partial\Omega \times \mathbb{R}, \mathbb{R})$ for some $0 < \epsilon < 1$, $h(x,\cdot) \in C^3(\mathbb{R},\mathbb{R})$ and $h(x,0) = 0$ for all $x \in \partial\Omega$, $u_\tau = u(x,t-\tau)$, in the equation (1.1), the current rate of change of the population $\frac{\partial u}{\partial t}$ depends on the population quantity at the past time point $t - \tau$. In system (1.1), a single bifurcation parameter, λ, controls both the internal and boundary reactions. When $\lambda = 0$, the equation becomes a flux-free diffusion equation with spatial memory.

In biology, the time delay $\tau \geq 0$ describes the averaged memory period, $u(x,t)$ represents the population density of a species at time t and location x, $m(x)$ is the intrinsic growth rate or the carrying capacity which can represent the situation of resource at x, in $d\nabla\cdot(u\nabla u_\tau)$ delay τ represents the averaged memory period, while in $\lambda u\left(m(x) - u_\tau\right)$ delay τ corresponds to the maturation time. In this paper, we focus on the case where memory and maturation delays are identical. The boundary conditions indicate that individuals reaching boundary $\partial\Omega$ are removed from the habitat at a rate determined by the current population density at that location.

Some scholars have studied the dynamical behaviors near the steady-state solutions of diffusion systems with Dirichlet boundary condition or Neumann boundary condition (for example, [7], [12], [18], [22]). However, many phenomena and processes exhibit nonlinear characteristics, such as the turbulence in fluids, nonlinear elasticity in materials, and variable rates in chemical reactions. Therefore, we incorporate more general nonlinear boundaries into the typical memory-inclusive diffusion systems. A lot of literature employs the center manifold method to investigate Hopf bifurcation (for example, [23], [24], [10]). However, the reduced equations through this method often remain high-dimensional, posing significant challenges for studying high-dimensional or even infinite-dimensional equations. In this paper, we utilize the Lyapunov-Schmidt procedure, aiming for a more precise and efficient characterization of system (1.1). Although studies such as An, Wang and Wang [1], Chen, Lou and Wei [6], and Ji and Wu [19] have investigated the impact of spatial memory on population dynamics, and Cantrell and Cosner [3], Cantrell, Cosner and Martínez [4], and Guo [14, 15] have explored the effects of nonlinear boundary conditions, few studies are devoted to their combined influences. Model (1.1) enables a thorough analysis, providing deeper insights into the dynamical behavior of models with both nonlinear boundaries and spatial memory.

The objective of this paper is to determine the set λ for which steady-state solutions exist, and to ascertain the uniqueness, stability, and Hopf bifurcation of these positive steady-state solutions based on the values of τ. Ji and Wu [20] explored the stability of steady-state solutions and Hopf bifurcations in model (1.1) with Neumann boundary condition. By incorporating nonlinear boundary condition, this paper facilitates a deeper understanding of the intrinsic mechanisms of