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Vector Fixed Point Theorem with Application to Systems of Nonlinear Elastic Beams Equations

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Abstract In this work, we establish a new existence and uniqueness of vector fixed point for a class of sum-type vector operators with some mixed monotone property in partially ordered product Banach spaces. The technique used is Thompson's part metric, and our goal is to extend and improve existing works in the scalar case vector case. As an application, we study the existence and uniqueness of solutions for systems of nonlinear singular fourth-order elastic beam equations with nonlinear boundary conditions.

Keywords Mixed monotone vector operators, Meir-Keeler type, systems of nonlinear elastic beams equations, Thompson metric, ε-chainable metric space

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1. Introduction

The theory of fixed points represents a growing field of research and development, intelligently combining different disciplines of knowledge such as geometry, topology and analysis. It is among the most powerful and fruitful tools of modern mathematics and can be considered a central subject of nonlinear analysis. In particular, when it comes to the solvability of a functional equation (whether it is a differential equation, a fractional differential equation, or an integral equation...), the problem is formulated in terms of finding a fixed point of a certain mapping. This theory has numerous applications, notably in biology, chemistry, economics and physics. For example, in [20], the authors demonstrated some fixed point theorems and used them to prove the solvability of certain fractional differential equations. It is noteworthy that these types of differential equations are frequently used in engineering sciences. See also [21], where the authors confirmed the effectiveness of fixed points theory in physics by applying it to the equation of motion.

Recently, in the context of the development of fixed point theory, the mixed monotone operators, which were first introduced in 1987 by Guo and Lakshmikantham [9], have provided some existence theorems for coupled fixed points for both continuous and discontinuous operators with coupled upper-lower solutions. They then proposed some applications to initial value problems of ordinary differential equations with discontinuous right-hand sides. As an extension of [9], an existence and uniqueness theorem was established in 1988 by Dajun Guo [8] for an operator

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$A : \overset{\circ}{P} \times \overset{\circ}{P} \to \overset{\circ}{P}$ under the following condition: there exists $0 \leq \alpha < 1$ such that $A(tx, t^{-1}y) \succeq t^{\alpha} A(x, y)$ for each $x, y \in \overset{\circ}{P}$ and $0 < t < 1$. These results have been developed and generalized in [22], where the authors prove new fixed point theorems for mixed monotone operators $A : P \times P \to P$ under the following conditions:

i) There exists $h \in P$ with $h \neq \theta$ such that $A(h, h) \in P_h$;

ii) For any $x, y \in P$ and $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1]$ such that $A(tx, t^{-1}y) \succeq \varphi(t) A(x, y)$.

Using the main results obtained, they give the local existence and uniqueness of positive solutions for the following nonlinear boundary value problems:

$$\begin{cases} -u''(t) + m^2 u(t) = \lambda f\left(t, u(t), u(t)\right), & 0 < t < 1, \\ u'(0) = u'(1) = 0. \end{cases}$$

The powerful role of the theory of fixed points has driven its extension in various directions. For example, one new direction involves extending the Banach contraction principle to metric spaces endowed with a partial order. See [5], where the authors established a fixed point theorem for a new class of mixed monotone operators, which are nearly asymptotically nonexpansive.

In this context, H. Wang et al. [18] obtained the existence and uniqueness of fixed point of the nonlinear sum operators $Ax + Bx + C(x, x)$, where A is an increasing α-concave (or sub-homogeneous) operator, B is a decreasing operator and C is a mixed monotone operator and they applied their results to a fractional differential equation. Thereafter, the authors [19] studied another abstract related to sum-type operator equation $A(x, x) + B(x, x) + Cx = x$, where A and B are two mixed monotone operators and C is an increasing operator, then the authors applied the result to a nonlinear fractional differential equation with multi-point fractional boundary conditions. In [14], Y. Sang et al. established the existence and uniqueness of solution for the operator equation $A(x, x) + B(x, x) + Cx + e = x$. The authors generalized the results obtained in [24] on the cone mappings to non-cone case. However, as far as we know, the fixed point results concerning vector operators with mixed monotone properties are still very limited. In [11], the authors established the existence and uniqueness a fixed point for the abstract vector operator equation $\Phi(x, y, x, y) = (A_1(x, x, y), A_2(x, y, y)) = (x, y)$, where $\Phi : P_h \times P_k \times P_h \times P_k \to P_h \times P_k$ has some mixed monotone properties with respect to the operators $A_1 : P_h \times P_h \times P_k \to P_h$ and $A_2 : P_h \times P_k \times P_k \to P_k$. Then they applied their results to obtain the positive solution for a system of nonlinear Neumman boundary value problems.

Being motivated by [11,15,23] and other works, we intend to study the existence and uniqueness of fixed point for the following system of operators, which we are going to consider it as an adequate vector operator with certain mixed monotone properties

$$\begin{aligned} A_1(x, x, y) + B_1(x, x, y) + e_1 = x, \\ A_2(x, y, y) + B_2(x, y, y) + e_2 = y. \end{aligned} \tag{1.1}$$

Then, we apply our result to show the existence and uniqueness of solutions to the system (4.2). The main result can be considered, to some extent, as a generalization of most of the results obtained in the cited references, in addition to being a transition from the scalar case to the vector case.