Journal of Nonlinear Modeling and Analysis
Volume 7, Number 2, April 2025, 739–763
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.739

Analysis of a Contact Problem Modeled by Hemivariational Inequalities in Thermo-Piezoelectricity

Mohammed Alaoui[1], El-Hassan Essoufi[1], Abdelhafid Ouaanabi[1,†] and Bouallala Mustapha[1,2]

Abstract We study a quasistatic contact problem from both variational and numerical perspectives, focusing on a thermo-piezoelectric body interacting with an electrically and thermally rigid foundation. The contact is modeled with a normal damped response and unilateral constraint for the velocity field, associated with a total slip-dependent version of Coulomb's law of dry friction. The electrical and thermal conditions on the contact surface are described by Clarke's subdifferential boundary conditions. We formulate the problem's weak form as a system combining a variational-hemivariational inequality with two hemivariational inequalities. Utilizing recent results in the theory of hemivariational inequalities, along with the fixed point method, we demonstrate the existence and uniqueness of the weak solution. Furthermore, we examine a fully discrete scheme for the problem employing the finite element method, and we establish error estimates for the approximate solutions.

Keywords Thermo-piezoelectric materials, friction, hemivariational inequality, fixed point argument, finite element method

MSC(2010) 74Fxx, 74M10, 58J20, 74M15, 47J22, 74S05.

1. Introduction

Recently, the study of contact problems between thermo-piezoelectric bodies, has garnered significant attention in both industrial and real-world scenarios, and remains an active area of research. These problems arise from the coupling of mechanical, electrical and thermal properties, In the literature, several mathematical results address thermo-piezoelectric contact problems. Some findings on mathematical modeling and variational analysis can be found in [1, 2, 8, 9, 13, 14]. Additionally, numerical schemes and their error estimates are discussed in [2, 7, 8, 13]. We extend these results to a quasistatic case by incorporating nonmonotone boundary conditions defined by Clarke's subdifferential and employing the principles of hemivariational inequalities.

†the corresponding author.
Email address: alaoui_fsts@yahoo.fr (M. Alaoui), e.h.essoufi@gmail.com (E-H. Essoufi), ouaanabi@gmail.com (A. Ouaanabi), bouallala-mustaphaan@gmail.com(M. Bouallala)
[1]Univ. Hassan 1, Laboratory MSDTE, 26000 Settat, Morocco
[2]Modeling and Combinatorics Laboratory, Department of Mathematics and Computer Science, Polydisciplinary Faculty, Cadi Ayyad University, B.P. 4162, Safi, Morocco

The theory of hemivariational inequalities was introduced in the early 1980s by Panagiotopoulos in [28, 29]. This theory is grounded in the properties of Clarke's subdifferential for locally Lipschitz functions, which may be nonconvex. These inequalities have been instrumental in describing and analyzing various problems in Mechanics, Physics, and Engineering Sciences, particularly in Contact Mechanics [10, 20].

The present paper introduces a new mathematical model for a quasistatic frictional contact between a thermo-piezoelectric body and an electrically and thermally conducting rigid foundation. The novelty of this model lies in the application of the normal damped response and unilateral constraint for the velocity field. The damper coefficient depends on the normal displacement, associated with a version of Coulomb's law of dry friction, in such a way that the friction bound depends on the total slip, and in modeling the electrical and thermal conditions on the contact surface using subdifferential boundary conditions involving nonconvex functionals. From a mathematical perspective, we demonstrate the well-posedness of the resulting model. To approximate the solution, we propose a fully discrete scheme and estimate the error between the numerical solution and the exact solution, achieving optimal order accuracy for the linear finite element method under additional regularity assumptions.

The rest of the paper is organized as follows. In Section 2, we present the model of a thermo-piezoelectric body in a quasistatic frictional contact with a conductive rigid foundation. In Section 3, we introduce the notation and assumptions for the problem's data and derive the variational formulation of the problem. Section 4 contains the existence and uniqueness proof for a weak solution to the problem. Finally, in Section 5, we propose a fully discrete scheme for the numerical solution, along with related error estimates and convergence results.

2. Problem statement

In the current section we present a classic formulation of the contact problem of a thermo-piezoelectric body with a thermally and electrically conducting rigid foundation in a quasistatic process.

We consider a thermo-piezoelectric body which initially occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary $\Gamma = \partial\Omega$. The body is acted upon by body forces of density f_0, volume electric charges of density q_0, and volume heat source term q_{th} on Ω. It is also subject to mechanical, electrical and thermal constraints at its boundary. To formulate these constraints we divide Γ into three measurable and disjoint parts Γ_1, Γ_2 and Γ_3 on one hand, such that $|\Gamma_1| > 0$, and we also consider a partition of $\Gamma_1 \cup \Gamma_2$ into two measurable and disjoint parts Γ_a and Γ_b on the other hand, such that $|\Gamma_a| > 0$. We assume that the body is clamped on Γ_1, the electrical potential vanishes on Γ_a and the temperature is zero on $\Gamma_1 \cup \Gamma_2$. We also assume that surface tractions of density f_2 act on Γ_2 and a surface electrical charge of density q_b is prescribed on Γ_b. Over the contact surface Γ_3, the body may come frictional contact with a conductive obstacle, the so called foundation, whose potential and temperature are assumed to be maintained at φ_F and θ_F, respectively.

We denote by $[0, T]$ the time interval of interest, where $T > 0$, and by $x \in \Omega \cup \Gamma$ and $t \in [0, T]$ the spatial and the time variable, respectively. Sometimes, we omit the explicit dependence of various functions on x and t. Moreover, we use Div and