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Existence of Weak Solutions for a Kind of Parabolic Steklov Problems

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Abstract Our focus in this study revolves around investigating the following parabolic problem

$$\begin{cases} u_t - \Delta u + u = 0 & \text{in } \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = g(u) & \text{on } \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

By using the Galerkin approximation and a family of potential wells, we obtain the existence of global solution and finite time blow-up under some suitable conditions. On the other hand, the results for asymptotic behavior of certain solutions with positive initial energy are also given.

Keywords Parabolic problem, global existence, blow-up, asymptotic behavior

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1. Introduction and preliminaries

The motivation for parabolic problems in the context of partial differential equations (PDEs) comes from various fields of physics, engineering and applied sciences. Parabolic equations model phenomena that evolve in time and space, where diffusion plays an important role, such as heat diffusion (heat equation), matter diffusion (diffusion equation), viscous fluid motion (Navier-Stokes equation), wave propagation in a dissipative medium, etc. (see [3, 7–10, 13, 17, 19, 21, 22]). The physical modeling of such equations often involves the numerical resolution of these equations using methods such as the finite difference method, the finite element method or the finite volume method. These methods discretize the continuous equations onto a spatial grid and solve the problem numerically to obtain an approximate solution that represents the physical behavior of the system under study (see [1, 2, 5, 15, 16]).

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In the present article, we mainly study the following Steklov parabolic problem

$$\begin{cases} u_t - \Delta u + u = 0 & \text{in } \ \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu} = g(u) & \text{on } \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x) & \text{in } \ \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with a smooth boundary $\partial\Omega$, and $g(u)$ satisfies the conditions as follows:

$$\text{(C)} \quad \begin{cases} g \in C^1 \text{ and } g(0) = g'(0) = 0, \\ g(u) \text{ is monotone, and is convex for } u > 0, \text{concave for } u < 0, \\ (p+1)G(u) \leqslant ug(u), |ug(u)| \leqslant \mu|G(u)|, \end{cases} \tag{1.2}$$

where

$$G(u) = \int_0^u g(s)\mathrm{d}s,$$

and

$$\begin{cases} 2 < p+1 \leqslant \mu < \infty & \text{if } N = 2, \\ 2 < p+1 \leqslant \mu \leqslant \frac{2(N-1)}{N-2} & \text{if } N \geqslant 3. \end{cases}$$

In the literature, there are several works dealing with Steklov-type parabolic problems (see [11,14,18]). For example, in [11] C. Enache has treated the following quasilinear initial-boundary value problem:

$$\begin{cases} u_t = \operatorname{div}(b(u)\nabla u) + f(u) & \text{in } \Omega, \ t > 0 \\ \frac{\partial u}{\partial n} + \kappa u = 0 & \text{on } \ \partial\Omega, \ t > 0, \\ u(x,0) = h(x) \geq 0 & \text{in } \Omega. \end{cases}$$

Under the suitable assumptions on the functions b, f and h, the author established a sufficient condition to guarantee the occurrence of the blow-up. Moreover, a lower bound for the blow-up time was obtained.

Also, L. E. Payne and P. W. Schaefer in [18] considered the heat equation subject to a nonlinear boundary condition, i.e.

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \ \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu} = f(u) & \text{on } \partial\Omega, \ t > 0, \\ u(x,0) = g(x) \geq 0 & \text{in } \ \Omega, \end{cases}$$

where Ω is a bounded smooth convex domain in $\mathbb{R}^3$ and f satisfies the condition

$$0 \leqslant f(s) \leqslant ks^{(n+2)/2}, \quad s > 0,$$

for some positive constants k and $n \geqslant 1$. By using a differential inequality technique, the authors determined a lower bound on the blow-up time for solutions of the heat equation when the solution explosion occurs. In addition, a sufficient condition which implies that blow-up does occur was determined.

In [14], A. Lamaizi et al. have considered the following problem:

$$\begin{cases} u_t - \Delta u + u = 0 & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = \lambda|u|^{p-1}u & \text{on } \partial\Omega \times (0,T), \\ u(x;0) = u_0(x) & \text{in } \Omega, \end{cases}$$