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Orbital Stability of the Sum of N Peakons for the CH-mCH Equation*

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Abstract This paper is concerned with a generalization of the modified Camassa-Holm equation with both cubic and quadratic nonlinearities (also known as the CH-mCH equation). We mainly prove the orbital stability of the train of peakons for the CH-mCH equation in energy space, using energy arguments and combining the method of orbital stability of a single peakon with the monotonicity of the local energy norm.

Keywords Camassa-Holm equation, CH-mCH equation, peakons, multi-peakons, orbital stability

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1. Introduction

In this paper, we consider the multi-peakon solutions of the following CH-mCH equation [15]

$$m_t + k_1((u^2 - u_x^2)m)_x + k_2(2u_x m + u m_x) = 0, t > 0, x \in \mathbb{R}, \tag{1.1}$$

where $m = u - u_{xx}$, k_1 and k_2 are two arbitrary constants, Eq. (1.1) is completely integrable and admits the Lax pair and bi-Hamiltonian structure [38]. The Cauchy problem and well-posedness were considered in [28].

Notice that when $k_1 = 0, k_2 = 1$, Eq. (1.1) reduces to the Camassa-Holm (CH) equation

$$m_t + 2u_x m + u m_x = 0, \quad m = u - u_{xx}, \tag{1.2}$$

which was derived as a model for shallow water waves [3], where $u(t,x)$ denotes the free surface above the flat bottom. Eq. (1.2) has many interesting properties: the existence of peaked solutions and multi-peakons [1, 3], wave-breaking phenomena [7–9] and geometric formulations [6]. Fuchssteiner and Fokas [16] first noted that Eq. (1.2) has a bi-Hamiltonian structure and hence infinitely many conservation laws. Camassa and Holm [3] obtained the single peakons of Eq. (1.1), which takes the form [30],

$$u(t,x) = c\varphi(x - ct) = ce^{-|x-ct|}, \quad c \in \mathbb{R}, \tag{1.3}$$

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and the multi-peakons

$$u(t,x)=\sum_{i=1}^{N}p_i(t)e^{-|x-q_i(t)|}, \tag{1.4}$$

where $p_i(t)$ and $q_i(t)$ satisfy the Hamiltonian system

$$\begin{cases} \dot{p}_i=\sum_{j\neq i}p_ip_j\mathrm{sign}(q_i-q_j)e^{-|q_i-q_j|}=-\frac{\partial H}{\partial q_i}, \\ \\ \dot{q}_i=\sum_j p_je^{-|q_i-q_j|}=\frac{\partial H}{\partial p_i}, \end{cases} \tag{1.5}$$

with the Hamiltonian

$$H=\frac{1}{2}\sum_{i,j=1}^{N}p_ip_je^{-|q_i-q_j|}. \tag{1.6}$$

Constantin and Strauss [11] proved orbital stability using energy as a Lyapunov function and basing on the conservation law of the CH equation. A variational approach for proving the orbital stability of the peakons was introduced by Constantin and Molinet [10]. The variational approach was extended to prove the orbital stability of the peakons for the other nonlinear wave equations [4, 17, 22, 25, 29, 33, 41]. Orbital stability of multi-peakon solutions was discussed by Dika and Molinet in [14].

When $k_1=1, k_2=0$, Eq. (1.1) reduces to the mCH\FORQ equation

$$m_t+\big((u^2-u_x^2)m\big)_x=0,\quad m=u-u_{xx}. \tag{1.7}$$

The orbital stability of the single peakons and the train of peakons for (1.7) was proved in [24] and [35], respectively. After that, Li [19] established the orbital stability of the peakons under $H^1\cap W^{1,4}$ norm.

We also introduce the gmCH equation proposed in [2]:

$$m_t+\big((u^2-u_x^2)^n m\big)_x=0,\quad m=u-u_{xx}, \tag{1.8}$$

where $n\geqslant 1$ is a positive integer. Eq. (1.8) becomes the fifth-order CH-type equation when $n=2$. The orbital stability of periodic peakons was examined by [32]. When $n=3$, Liu [26, 27] investigated the orbital stability of a higher-order nonlinear modified Camassa-Holm equation with peakons and multi-peakons. The local well-posedness and blow-up mechanism of Eq. (1.8) have been discussed in [39]. The orbital stability of peakons for Eq. (1.8) has been demonstrated by Guo et al. in [18]. Deng and Chen [13] have also proved the orbital stability of the sum of N peakons. Recently, a variety of CH-type equations have been explored, including the mCH-Novikov equation [31], the generalized cubic-quintic Camassa-Holm type equation [37], the b-family of FORQ/MCH equations [40], etc. Orbital stability of the single peakons and multi-peakons for the mCH-Novikov equation and the generalized cubic-quintic Camassa-Holm type equation has been proved by [5, 12, 36, 37]. For the Camassa-Holm-type equations, different wave profiles of φ for different types of phase orbits were classified using dynamical system theory in [20, 21].

More generally, Eq. (1.1) also has single peakons, periodic peakons and multi-peakons. Its orbital stability has been proved by Liu et al. in [23]. In this paper, we prove that the multi-peakons of Eq. (1.1) are orbitally stable in energy space.