Journal of Nonlinear Modeling and Analysis
Volume 7, Number 3, June 2025, 840–867
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.840

# Optimal and Stability Analysis of the Co-infections Disease Mathematical Model

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**Abstract** A co-infection of a human or a pig with human influenza or COVID-19 strains and H5N1 strain may result in a pandemic strain, causing a widespread deadly pandemic. In this paper, we consider a new class of co-infections disease epidemic models for a rapid and slow virus. We study the transmission threshold by analyzing the basic reproduction number. The equilibrium points for the model are derived, and their local stability is analyzed with suitable assumptions on the model parameters. Understanding the model parameters is one of the prime subjects in this research work. Therefore, the sensitivity of essential parameters is investigated. Moreover, the optimal control problem for the proposed model is considered, and first-order optimality conditions are derived. Finally, numerical simulations indicate the effects of the model's basic reproduction number and control variables.

**Keywords** Co-infections model, stability analysis, basic reproduction number, optimal control

**MSC(2010)** 34D05, 34D20, 49J15, 92C60.

## 1. Introduction

Time evolution and the rapid spread of deadly viruses threaten human life and global economic growth. Therefore, understanding virus transmission is excellent attention to take precautions to control the disease. Recently, human beings suffers a lot from similar virus transmissions. The spread of the virus around the globe agitates its living organisms. For example, COVID-19 ruled out human life for a year, both health and wealth-wise. The spread of COVID-19 has been modeled and discussed by many researchers in their recent articles, for example, see [1, 10, 19, 20] and also the references therein. In particular, a patient with other health ailments (e.g., asthma, chronic obstructive pulmonary disease (COPD), pulmonary fibrosis, pneumonia, lung cancer, diabetes of type 1, human immunodeficiency virus (HIV), etc.,) suffered a lot in this COVID-19 outbreak.

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On the other hand, some virus for example asthma, chronic obstructive pulmonary disease (COPD), pulmonary fibrosis, pneumonia, lung cancer, diabetes of type 1 and HIV are virus that damages the cells in your immune system and weakens your ability to fight everyday infections and diseases. Some of the virus are not fully curable, but we can make patients lead their daily lives without any troubles. However, for more detail, we refer the interested readers to [2–4, 9] and the references therein. A mathematical model studies the transmission of two or more diseases called a co-infection model. Therefore, a good understanding of both the virus transmission and outbreak is important to investigate. Hence, we propose a co-infection diseases mathematical model in this work. For more references of co-infection diseases mathematical model, we refer the readers to [7, 11, 12, 14] and the references therein.

Coinfection is the process of infection of a single host with two or more pathogen variants (strains) or with two or more distinct pathogen species. Coinfection with multiple pathogen strains is particularly common in HIV, but it occurs in many other diseases. Coinfection with multiple pathogen species is also thought to be a very common occurrence. Particularly widely distributed combinations are HIV and tuberculosis, HIV and hepatitis, HIV and malaria, and others. Coinfection is of significant importance because it may have negative effect both on the health of the coinfected individuals as well as on the public health in general. For instance, a coinfection of a human or a pig with human influenza strain and H5N1 strain may result in a pandemic strain, causing a widespread deadly pandemic. Among rapid virus such as COVID-19 or influenza affected patients, few of them are asymptomatic. Further, some are exposed to this virus patients are in quarantine. Those classes are also included in the following co-infection model. The population of the model includes twelve classes: $S$ denotes the the susceptible individuals, $E_1$ and $E_2$ represent the exposed individuals of first (rapid virus) and Second (slow virus) respectively. $I_1$ for infected individuals of first virus, $I_2$ for infected individuals of second virus, and $I_3$ represents the individuals who are infected by both of the infections. The variable $A$ denotes the individuals who are asymptomatic of first virus. $Q$ denotes individuals who are in quarantine for first virus. $R_1$ denotes the individuals who are recovered from first virus. $R_2$ represents the individuals who are recovered from second virus, and $R_3$ represents the individuals who are recovered from both infections. Finally, $D$ denotes individuals who died due to infection. Here the recovery of second virus individuals represents the individuals who can lead their life without any interception of the infections. To model coinfection, we need to introduce a new dependent variable, namely $I_3(t)$, the number of coinfected individuals in the population. The model again is built on the basis of the competitive exclusion model (1.1), but with a coinfected class $I_3$. From the model flow diagram of the co-infections disease mathematical epidemic model Fig. 1, we derive the following system of nonlinear differential equations

$$\left.\begin{aligned}
\frac{dS}{dt} &= \Lambda - \beta_1 S\left(I_1 + A\right) - p\beta_3 S I_3 - \beta_2 S I_2 - (1-p)\beta_3 S I_3 + r_1 R_1 + r_2 R_2 + r_3 R_3 - \mu S,\\
\frac{dE_1}{dt} &= \beta_1 S\left(I_1 + A\right) + p\beta_3 S I_3 - \tau_1 E_1 - \rho\delta E_1 + \beta_1 I_1 R_2 + \beta_3 I_3 R_2 - \mu E_1,\\
\frac{dE_2}{dt} &= \beta_2 S I_2 + (1-p)\beta_3 S I_3 - \tau_2 E_2 + \beta_2 I_2 R_1 + \beta_3 I_3 R_1 - \mu E_2,\\
\frac{dI_1}{dt} &= \tau_1 E_1 - \theta_1 b I_1 I_2 - b I_1 I_3 - \sigma I_1 - \mu I_1,\\
\frac{dI_2}{dt} &= \tau_2 E_2 - \theta_2 b I_1 I_2 - b I_2 I_3 - \gamma_3 I_2 - d_2 I_2 - \mu I_2,\\
\frac{dI_3}{dt} &= \theta_1 b I_1 I_2 + b I_1 I_3 + \theta_2 b I_1 I_2 + b I_2 I_3 + \theta_3 b A I_2 + b A I_3 - d_3 I_3 - \gamma_4 I_3 - \mu I_3,\\
\frac{dA}{dt} &= \sigma\alpha I_1 - \rho A - \gamma_1 A - \theta_3 b A I_2 - b A I_3 - \mu A,\\
\frac{dQ}{dt} &= \sigma(1-\alpha) I_1 + \rho A - \gamma_2 Q - d_1 Q + \delta\rho E_1 - \mu Q,\\
\frac{dR_1}{dt} &= \gamma_1 A + \gamma_2 Q - r_1 R_1 - \beta_2 I_2 R_1 - \beta_3 I_3 R_1 - \mu R_1,\\
\frac{dR_2}{dt} &= \gamma_3 I_2 - r_2 R_2 - \beta_1 I_1 R_2 - \beta_3 I_3 R_2 - \mu R_2,\\
\frac{dR_3}{dt} &= \gamma_4 I_3 - r_3 R_3 - \mu R_3,\\
\frac{dD}{dt} &= d_1 Q + d_2 I_2 + d_3 I_3,
\end{aligned}\right\}, \tag{1.1}$$

where $\theta_1 + \theta_2 + \theta_3 = 1$, $1 > \alpha$ and the initial values are given as,

$$S(0) = S_0,\ E_1(0) = E_{10},\ E_2(0) = E_{20},\ I_1(0) = I_{10},\ I_2(0) = I_{20},\ I_3(0) = I_{30},$$

$$A(0) = A_0,\ Q(0) = Q_0,\ R_1(0) = R_{10},\ R_2(0) = R_{20},\ R_3(0) = R_{30},\ D(0) = D_0.$$

The parameters used in the model are defined in Table 1. All the parameters used in the model (1.1) are positive. Before starting the main results of the work, we recall some literature on the co-infection model and its related study. Recently a mathematical model for Buruli ulcer and Cholera diseases was proposed, and stability analysis was performed in [22]. A model for HIV and hepatitis C virus (HCV) co-infection is studied, and the reproduction numbers and the local and global stability of the disease-free equilibria are derived in [5]. A fractional-order model for the co-infection of HIV and tuberculosis (TB), in the presence of multi-drug resistant TB strains (MDR-TB), and treatment for both diseases are analyzed in [15]. A generalized simple mathematical model of HIV-COVID-19 together is studied in [8]. A new mathematical model for dual variants of COVID-19 and HIV co-infection is presented and analyzed in [13]. A model for COVID-19 and HIV/AIDS is proposed to assess the impact of COVID-19 on HIV dynamics and vice-versa in [16]. However, all the above models are considered with some restrictions in the recovery class and do not include the quarantine and asymptomatic classes.

The remaining paper is arranged as follows: In section 2, we discuss the solvability and boundedness of the proposed model. Further, the basic reproduction

**Table 1.** Definition of parameter used in the model (1.1)

| Parameters | Definition |
|---|---|
| $\Lambda$ | recruitment rate |
| $\beta_1$ | infection rate of pathogen 1 |
| $\beta_2$ | infection rate of pathogen 2 |
| $\beta_3$ | infection rate of both the pathogen |
| $p$ | probability rate of infection from $I_3$ which transform $S$ to $I_1$ |
| $r_1$ | rate of person who again going to $S$ from $R_1$ |
| $r_2$ | rate of person who again going to $S$ from $R_2$ |
| $r_3$ | rate of person who again going to $S$ from $R_3$ |
| $\tau_1$ | per capita rate at which the exposed individuals of panthogen 1 become infections |
| $\tau_2$ | per capita rate at which the exposed individuals of panthogen 2 become infections |
| $\rho$ | testing rate of people with mild or no symptoms at the time |
| $\delta$ | probability of detecting infection by testing in $E_1$ |
| $\theta_1$ | transmission rate of a individual from $I_1$ to $I_3$ |
| $\theta_2$ | transmission rate of a individual from $I_2$ to $I_3$ |
| $\theta_3$ | transmission rate of a individual from $A$ to $I_3$ |
| $b$ | co-infection rate |
| $\sigma$ | inverse of the time from infectiousness onset to possible symptoms onset |
| $\gamma_1$ | recovery rate of asymptomatic individuals |
| $\gamma_2$ | recovery rate of quarantine individuals |
| $\gamma_3$ | recovery rate of second virus individuals |
| $\gamma_4$ | recovery rate of co-infection individuals |
| $d_1$ | death rate due to first virus in quarantine |
| $d_2$ | death rate due to second virus |
| $d_3$ | death rate due both the virus |
| $\alpha$ | proportion of asymptomatic infection |
| $\mu$ | natural death rate |

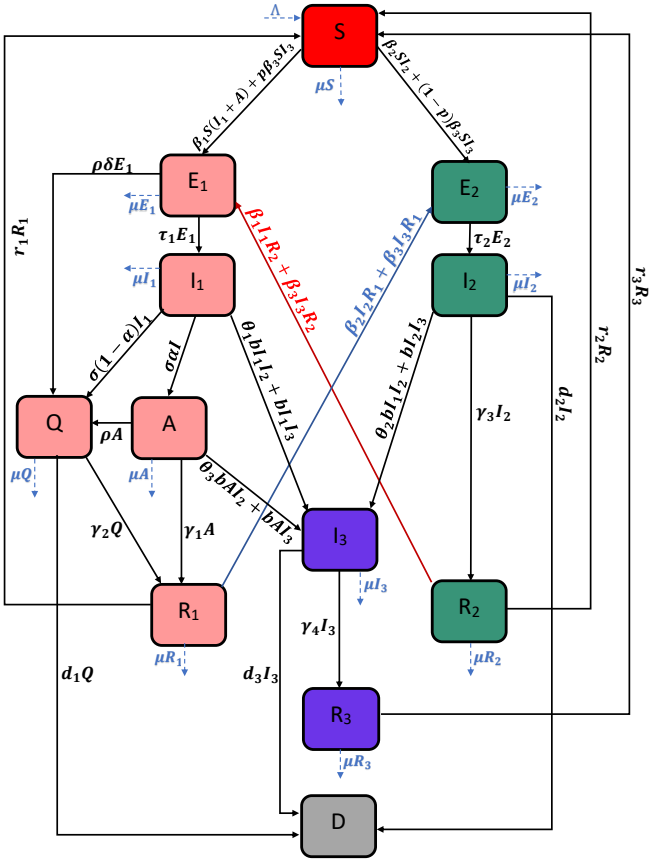


**Figure 1.** Schmatic diagram for coinfection model.

number for the model (1.1) and the local stability of all possible equilibrium points are studied in section 3. The sensitivity of model parameters are discussed in section 4. The optimal control problem is proposed and analyzed in section 5. Computational results are provided in section 6.

## 2. Solvability and boundedness of solution of coinfection model

In this section, we prove the existence and uniqueness of the solution of the model (1.1). Further, the non-negative and boundedness of the solution are also analyzed for the model (1.1). We rewrite the model (1.1) as follows,

$$\frac{dX}{dt} = F(X(t)), \quad t \in (0, T], \tag{2.1}$$

with a initial condition $X(0) = X_0$. Here

$$X(t) = [S(t)\ E_1(t)\ E_2(t)\ I_1(t)\ I_2(t)\ I_3(t)\ A(t)\ Q(t)\ R_1(t)\ R_2(t)\ R_3(t)\ D(t)]^T,$$

and a non-linear function is defined as,

$$F(X(t)) = \big[F_1(X(t)), F_2(X(t)), F_3(X(t)), F_4(X(t)), F_5(X(t)), F_6(X(t)), F_7(X(t)), F_8(X(t)), F_9(X(t)), F_{10}(X(t)), F_{11}(X(t)), F_{12}(X(t))\big]^T,$$

where

$$\begin{aligned}
F_1(X(t)) &= \Lambda - \beta_1 S\,(I_1 + A) - \beta_2 S I_2 - \beta_3 S I_3 + r_1 R_1 + r_2 R_2 + r_3 R_3 - \mu S,\\
F_2(X(t)) &= \beta_1 S\,(I_1 + A) + p\beta_3 S I_3 - \tau_1 E_1 - \rho\delta E_1 + \beta_1 I_1 R_2 + \beta_3 I_3 R_2 - \mu E_1,\\
F_3(X(t)) &= \beta_2 S I_2 + (1-p)\beta_3 S I_3 - \tau_2 E_2 + \beta_2 I_2 R_1 + \beta_3 I_3 R_1 - \mu E_2,\\
F_4(X(t)) &= \tau_1 E_1 - \theta_1 b I_1 I_2 - b I_1 I_3 - \sigma I_1 - \mu I_1,\\
F_5(X(t)) &= \tau_2 E_2 - \theta_2 b I_1 I_2 - b I_2 I_3 - \gamma_3 I_2 - d_2 I_2 - \mu I_2,\\
F_6(X(t)) &= \theta_1 b I_1 I_2 + b I_1 I_3 + \theta_2 b I_1 I_2 + b I_2 I_3 + \theta_3 b A I_2 + b A I_3 - d_3 I_3 - \gamma_4 I_3 - \mu I_3,\\
F_7(X(t)) &= \sigma\alpha I_1 - \rho A - \gamma_1 A - \theta_3 b A I_2 - b A I_3 - \mu A,\\
F_8(X(t)) &= \sigma(1-\alpha) I_1 + \rho A - \gamma_2 Q - d_1 Q + \delta\rho E_1 - \mu Q,\\
F_9(X(t)) &= \gamma_1 A + \gamma_2 Q - r_1 R_1 - \beta_2 I_2 R_1 - \beta_3 I_3 R_1 - \mu R_1,\\
F_{10}(X(t)) &= \gamma_3 I_2 - r_2 R_2 - \beta_1 I_1 R_2 - \beta_3 I_3 R_2 - \mu R_2,\\
F_{11}(X(t)) &= \gamma_4 I_3 - r_3 R_3 - \mu R_3,\\
F_{12}(X(t)) &= d_1 Q + d_2 I_2 + d_3 I_3.
\end{aligned}$$

**Lemma 2.1.** *Suppose* $\bar{X}(t) = \big[\bar{S}(t)\ \bar{E}_1(t)\ \bar{E}_2(t)\ \bar{I}_1(t)\ \bar{I}_2(t)\ \bar{I}_3(t)\ \bar{A}(t)\ \bar{Q}(t)\ \bar{R}_1(t)\ \bar{R}_2(t)\ \bar{R}_3(t)\ \bar{D}(t)\big]^T$ *and consider the state variables* $X$ *and* $\bar{X}$ *have a upper bound* $\Phi$*, then it satisfies*

$$|F(X(t))| - |F(\bar{X}(t))| < K|X - \bar{X}|, \tag{2.2}$$

*for some* $K > 0$.

**Proof.** Consider for the first compartment of $F(X(t))$ and evaluate as follows to get,

$$\begin{aligned}
&|F_1(X) - F_1(\bar{X})|\\
=&|-\beta_1 S\,(I_1 + A) - \beta_3 S I_3 - \beta_2 S I_2 + r_1 R_1 + r_2 R_2 + r_3 R_3 - \mu S\\
&-(-\beta_1 \bar{S}\,(\bar{I}_1 + \bar{A}) - \beta_3 \bar{S}\bar{I}_3 - \beta_2 \bar{S}\bar{I}_2 + r_1\bar{R}_1 + r_2\bar{R}_2 + r_3\bar{R}_3 - \mu\bar{S})|\\
\le&|\beta_1(S(I_1 + A) - \bar{S}\,(\bar{I}_1 + \bar{A}))| + |\beta_2(SI_2 - \bar{S}\bar{I}_2)| + |\beta_3(SI_3 - \bar{S}\bar{I}_3)|\\
&+|r_1(R_1 - \bar{R}_1)| + |r_2(R_2 - \bar{R}_2)| + |r_3(R_3 - \bar{R}_3)| + |\mu(S - \bar{S})|\\
\le&L_1(|S - \bar{S}| + |I_1 - \bar{I}_1| + |I_2 - \bar{I}_2| + |I_3 - \bar{I}_3| + |A - \bar{A}| + |R_1 - \bar{R}_1|\\
&+|R_2 - \bar{R}_2| + |R_3 - \bar{R}_3|)\\
\le&L_1|X - \bar{X}|,
\end{aligned} \tag{2.3}$$

where $L_1 = 2\Phi(2\beta_1 + \beta_2 + \beta_3) + r_1 + r_2 + r_3 + \mu$. As similar as in the above estimate, for all $F_i(X(t))$, we get

$$|F_i(X(t)) - F_i(\bar{X}(t))| \le L_i|X(t) - \bar{X}(t)| \quad (i = 2, 3, \cdots, 12). \tag{2.4}$$

Here

$$L_2 = 2\Phi(3\beta_1 + (p+1)\beta_3) + \tau_1 + \rho\delta + \mu,$$
$$L_3 = 2\Phi(\beta_2 + (1+p)\beta_3 + \beta_2 + \beta_3) + \tau_2 + \mu,$$
$$L_4 = \tau_1 + 2\Phi(\theta_1 b + b) + \sigma + \mu, \qquad L_5 = 2\Phi(\theta_2 b + b) + \tau_2 + \gamma_3 + d_2 + \mu,$$
$$L_6 = 2\Phi(b(\theta_1 + \theta_2 + \theta_3 + 3)) + d_3 + \gamma_4 + \mu, \quad L_7 = \sigma\alpha + \rho + \gamma_1 + \mu + 2\Phi(b(1+\theta_3)),$$
$$L_8 = \sigma(1-\alpha) + \rho + \gamma_2 + d_1 + \delta\rho + \mu, \qquad L_9 = \gamma_1 + \gamma_2 + r_1 + 2\Phi(\beta_2 + \beta_3) + \mu,$$
$$L_{10} = \gamma_3 + r_2 + 2\Phi(\beta_1 + \beta_3) + \mu, \qquad L_{11} = \gamma_4 + r_3 + \mu,$$
$$L_{12} = d_1 + d_2 + d_3.$$

The above inequality leads to,

$$|F(X(t))| - |F(\bar{X}(t))| < K|X - \bar{X}|, \tag{2.5}$$

where $K = \max\{L_1, L_2, \cdots, L_{12}\} > 0$. □

**Theorem 2.1.** *Suppose there exists a solution to the model* (1.1) *with the initial values belongs to* $\mathbb{R}^{12}_+$, *with* $S_0 > \dfrac{\Lambda}{\mu}$, *and all the model parameters being positive. Then that solution set* $X(t)$ *always remains in* $\mathbb{R}^{12}_+$. *Further, the solution of the model* (1.1) *is always bounded.*

**Proof.** First, the positivity of the solution is proved as follows: Consider the trajectory of the solution along $S$-axis, it means that, $E_1 = E_2 = I_1 = I_2 = I_3 = A = Q = R_1 = R_2 = R_3 = 0$ and $S(0) = S_0 > \dfrac{\Lambda}{\mu}$. Then from the first equation of (1.1), we have

$$\frac{dS}{dt} = \Lambda - \mu S$$

with $S(0) = S_0$. Solving the above differential equation, we get

$$S(t) = \frac{\Lambda - \exp(-\mu t)(\Lambda - \mu S_0)}{\mu} > 0.$$

Similarly, moving along the respective all other axes, that is considering variables are zero except the respective variable. We get all other state variables that are also non-negative. Now, consider $t^* > 0$ such that

$$S(t^*) = 0,\ E_1(t^*) > 0,\ E_2(t^*) > 0,\ I_1(t^*) > 0,\ I_2(t^*) > 0,\ I_3(t^*) > 0,\ A(t^*) > 0,$$
$$Q(t^*) > 0,\ R_1(t^*) > 0,\ R_2(t^*) > 0,\ R_3(t^*) > 0, \text{ and } S(t) < S(t^*).$$

On this plane,

$$\left.\frac{dS}{dt}\right|_{t=t^*} = \Lambda + r_1 R_1(t^*) + r_2 R_2(t^*) + r_3 R_3(t^*) > 0.$$

Using the generalized mean value theorem [18], we get

$$S(t) - S(t^*) = S'(\tau)(t - t^*),\ \tau \in (t^*, t],$$

from the above, $S(t) > S(t^*)$, and this contradicts our assumption for $t^*$. This shows, the solution $S(t)$ is positive $\forall\ t \geq 0$. Similar arguments on other variables lead to

$$\begin{aligned}
\left.\frac{dE_1}{dt}\right|_{t=t^*} &= \beta_1 S\left(I_1 + A\right) + p\beta_3 S I_3 + \beta_1 I_1 R_2 + \beta_3 I_3 R_2 \geq 0,\\
\left.\frac{dE_2}{dt}\right|_{t=t^*} &= \beta_2 S I_2 + \beta_3 S I_3 + \beta_2 I_2 R_1 + \beta_3 I_3 R_1 \geq 0,\\
\left.\frac{dI_1}{dt}\right|_{t=t^*} &= \tau_1 E_1 \geq 0,\\
\left.\frac{dI_2}{dt}\right|_{t=t^*} &= \tau_2 E_2 \geq 0,\\
\left.\frac{dI_3}{dt}\right|_{t=t^*} &= \theta_1 b I_1 I_2 + b I_1 I_3 + \theta_2 b I_1 I_2 + b I_2 I_3 + \theta_3 b A I_2 + b A I_3 \geq 0,\\
\left.\frac{dA}{dt}\right|_{t=t^*} &= \sigma\alpha I_1 \geq 0,\\
\left.\frac{dQ}{dt}\right|_{t=t^*} &= \sigma I_1 + \rho A + \delta\rho E_1 \geq 0,\\
\left.\frac{dR_1}{dt}\right|_{t=t^*} &= \gamma_1 A + \gamma_2 Q \geq 0,\\
\left.\frac{dR_2}{dt}\right|_{t=t^*} &= \gamma_3 I_2 \geq 0,\\
\left.\frac{dR_3}{dt}\right|_{t=t^*} &= \gamma_4 I_3 \geq 0,\\
\left.\frac{dD}{dt}\right|_{t=t^*} &= d_1 Q + d_2 I_2 + d_3 I_3 \geq 0.
\end{aligned} \tag{2.6}$$

Since the function $F_i(X(t)) \in C[a,b]$, and the state variables are also a continuous functions in the interval $[a,b]$, which shows that the solution of (2.6) starts from initial values belonging to $\mathbb{R}^{12}_+$ keep increasing and remains in $\mathbb{R}^{12}_+$.

Finally, we prove the boundedness of solutions of (1.1). Suppose that the total population of the model (1.1), is

$N(t)$
$= S(t)+E_1(t)+E_2(t)+I_1(t)+I_2(t)+I_3(t)+A(t)+Q(t)+R_1(t)+R_2(t)+R_3(t)+D(t).$
Then,

$$\begin{aligned}
\frac{dN}{dt} &\leq \Lambda - \mu N,\\
N(t) &\leq N(0)e^{-\mu t} + \frac{\Lambda}{\mu}.
\end{aligned}$$

This completes the proof of the model. □

From the above theorem, we can define the following domain for the solutions of (1.1),

$$\Omega = \left\{ X(t) \in \mathbb{R}^{12}_+ | N(t) \leq \frac{\Lambda}{\mu} + \epsilon,\ \text{for}\ \epsilon\ >\ 0 \right\}. \tag{2.7}$$

**Theorem 2.2.** *Suppose that* $X_0 \in \mathbb{R}^{12}_+$, *with* $S(0) > \dfrac{\Lambda}{\mu}$ *and the function* $F_i(t, X(t))$ $i = 1, \cdots, 12$ *satisfies the Lipschitz condition:*

$$\|F_i(t, X_1) - F_i(t, X_2)\| \leq C_i \|X_1 - X_2\|, \quad (i = 1, \cdots, 12). \tag{2.8}$$

*In the above* $C_i's$ *are positive constants and* $(t, X_1)$ & $(t, X_2)$ *belongs to the domain* $\Omega$ *where* $\Omega$ *defined in* (2.7) *Then there exists a unique continuous vector X(t) as a solution of the model* (1.1).

**Proof.** From the Lemma (2.1), we easily say that

$$\|F_i(t, X_1) - F_i(t, X_2)\| \leq C_i \|X_1 - X_2\|, \quad (i = 1, \cdots, 12),$$

where $C_i = L_i \quad (i = 1, 2, \cdots, 12)$ is defined in Lemma 2.1. It shows that the functions $F_i(t, X(t))$ $i = 1, \cdots, 12$ satstifies the uniform Lipschitz continuity with respect to state variables. This shows that the existence and the uniqueness of the solution of the model (1.1). □

# 3. Stability analysis

In this section, first, we define the disease-free equilibrium point for the model (1.1). Then, the basic reproduction number for the considered co-infection model is calculated. Further, all other possible equilibrium points are provided, and the local stability of all the equilibrium points are discussed. Finally, the necessary conditions to attend to global stability of disease-free equilibrium point is derived.

## 3.1. Basic reproduction number

Now, we start our computations for basic reproduction number by using the method proposed in [21]. We know that the disease-free equilibrium point of $(EP_0)$ to the model (1.1) is as follows:

$$EP_0 = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right).$$

Next, find the basic reproduction number for the model (1.1) using the following method. In order to do this, first, separate the newly infected term from all other evolving terms. Let $\mathcal{F}$ represent the rate at which new infections emerge in the compartments. $\mathcal{V}$ represents the rate at which the spread of infection from one compartment to another.

Here, $\mathcal{F}$ and $\mathcal{V}$ for the model (1.1) is given as follows:

$$\mathcal{F} = \begin{pmatrix} \beta_1 S\left(I_1 + A\right) + p\beta_3 S I_3 + \beta_1 I_1 R_2 + \beta_3 I_3 R_2 \\ \beta_2 S I_2 + (1-p)\beta_3 S I_3 + \beta_2 I_2 R_1 + \beta_3 I_3 R_1 \\ -\theta_1 b I_1 I_2 - b I_1 I_3 \\ -\theta_2 b I_1 I_2 - b I_2 I_3 \\ \theta_1 b I_1 I_2 + b I_1 I_3 + \theta_2 b I_1 I_2 + b I_2 I_3 + \theta_3 b A I_2 + b A I_3 \\ -\theta_3 b A I_2 - b A I_3 \\ 0 \end{pmatrix},$$

$$\mathcal{V} = \begin{pmatrix} \tau_1 E_1 + \rho\delta E_1 + \mu E_1 \\ \tau_2 E_2 + \mu E_2 \\ -\tau_1 E_1 + \sigma I_1 + \mu I_1 \\ -\tau_2 E_2 + \gamma_3 I_2 + d_2 I_2 + \mu I_2 \\ d_3 I_3 + \gamma_4 I_3 + \mu I_3 \\ -\sigma\alpha I_1 + \rho A + \gamma_1 A + \mu A \\ -\sigma(1-\alpha) I_1 - \rho A + \gamma_2 Q + d_1 Q - \delta\rho E_1 + \mu Q \end{pmatrix}.$$

So, we define the Jacobian matrix for $\mathcal{F}$ and $\mathcal{V}$ at $E_0$. It is denoted as $\mathbf{F}$ and $\mathbf{V}$ respectively as in the following form:

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & \frac{\beta_1\Lambda}{\mu} & 0 & \frac{p\beta_3\Lambda}{\mu} & \frac{\beta_1\Lambda}{\mu} & 0 \\ 0 & 0 & 0 & \frac{\beta_2\Lambda}{\mu} & (1-p)\frac{\beta_3\Lambda}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\mathbf{V} = \begin{pmatrix} \tau_1 + \rho\delta + \mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau_2 + \mu & 0 & 0 & 0 & 0 & 0 \\ -\tau_1 & 0 & \sigma + \mu & 0 & 0 & 0 & 0 \\ 0 & -\tau_2 & 0 & \gamma_3 + d_2 + \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_4 + d_3 + \mu & 0 & 0 \\ 0 & 0 & -\sigma\alpha & 0 & 0 & \rho + \gamma_1 + \mu & 0 \\ -\delta\rho & 0 & -\sigma(1-\alpha) & 0 & 0 & -\rho & \gamma_2 + d_1 + \mu \end{pmatrix}.$$

Hence, the basic reproduction number $R_0$ of (1.1) is the spectral radius of the next generation matrix $\mathbf{F}\mathbf{V}^{-1}$ which is given as follows:

$$R_0 = \rho\left(\mathbf{F}\mathbf{V}^{-1}\right) = \max\left\{\mathfrak{R}_1, \mathfrak{R}_2\right\}, \tag{3.1}$$

where

$$\begin{aligned} \mathfrak{R}_1 &= \frac{\beta_1\tau_1\Lambda\left(\rho + \gamma_1 + \mu + \sigma\alpha\right)}{\mu\left(\rho + \gamma_1 + \mu\right)\left(\tau_1 + \rho\delta + \mu\right)\left(\sigma + \mu\right)}, \\ \mathfrak{R}_2 &= \frac{\beta_2\tau_2\Lambda}{\mu\left(\tau_2 + \mu\right)\left(\gamma_3 + d_2 + \mu\right)}. \end{aligned} \tag{3.2}$$

Here, $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are threshold parameters representing the average number of secondary infection cases produced by a single infectious individual of pathogen 1

(COVID-19) and pathogen 2 (HIV), respectively. Further, $\mathfrak{R}_j,\ (j=1,2)$ are called as the basic reproduction number of the pathogen $j,\ (j=1,2)$.

For the model (1.1), we have only four equilibrium points as follows:

1. $EP_0 = \left(\frac{\Lambda}{\mu}, 0,0,0,0,0,0,0,0,0,0\right),$
2. $EP_1 = \left(S^{01}, E_1^{01}, 0, I_1^{01}, 0, 0, A^{01}, Q^{01}, R_1^{01}, 0, 0\right),$
3. $EP_2 = \left(S^{02}, 0, E_2^{02}, 0, I_2^{02}, 0, 0, 0, 0, R_2^{02}, 0\right),$
4. $EP_3 = \left(S^*, E_1^*, E_2^*, I_1^*, I_2^*, I_3^*, A^*, Q^*, R_1^*, R_2^*, R_3^*\right),$

where

$$S^{01} = \frac{\Lambda}{\mu}\frac{1}{\mathfrak{R}_1}, \qquad E_1^{01} = \frac{\sigma+\mu}{\tau_1} I_1^{01},$$

$$I_1^{01} = \frac{\Lambda}{\beta_1 S^{01}\left(1+\frac{\sigma\alpha}{\rho+\gamma_1+\mu}\right) - \gamma_1\mathfrak{A}}\left(1-\frac{1}{\mathfrak{R}_1}\right),$$

$$Q^{01} = \mathfrak{B} I_1^{01}, \qquad R_1^{01} = \mathfrak{A} I_1^{01}, \qquad A^{01} = \frac{\sigma\alpha}{\rho+\gamma_1+\mu} I_1^{01},$$

$$S^{02} = \frac{\Lambda}{\mu}\frac{1}{\mathfrak{R}_2}, \qquad E_2^{02} = \frac{\gamma_3+d_2+\mu}{\tau_2} I_2^{02},$$

$$I_2^{02} = \frac{\Lambda}{\beta_2 S^{02} - \frac{r_2\gamma_3}{r_2+\mu}}\left(1-\frac{1}{\mathfrak{R}_2}\right),$$

$$R_2^{02} = \frac{\gamma_3}{\gamma_2+\mu} I_2^{02},$$

$$\mathfrak{A} = \frac{1}{\gamma_1+\mu}\left\{\frac{\gamma_1\sigma\alpha}{\rho+\gamma_1+\mu} + \frac{\gamma_2}{\gamma_2+d_1+\mu}\left[\sigma(1-\alpha) + \frac{\rho\sigma\alpha}{\rho+\gamma_1+\mu} + \frac{\delta\rho(\sigma+\mu)}{\tau_1}\right]\right\},$$

$$\mathfrak{B} = \frac{1}{\gamma_2+d_1+\mu}\left[\sigma(1-\alpha) + \frac{\rho\sigma\alpha}{\rho+\gamma_1+\mu} + \frac{\delta\rho(\sigma+\mu)}{\tau_1}\right].$$

$EP_1$ is a equilibrium point where the system has only COVID-19. $EP_2$ is a equilibrium point where there is only HIV there is no COVID-19. Final, the endemic equilibrium point is denoted as $EP_3$, where both the infections are positive.

## 3.2. Local stability

In this subsection, the local stability for the equilibrium points are analyzed using the Routh Hurwitz criteria [17].

**Theorem 3.1.** *The disease free equilibrium point $EP_0$ of the model* (1.1) *is locally stable, if $R_0 < 1$ with $\beta_1\tau_1\Lambda < \mu(\tau_1+\rho\delta+\mu)(\sigma+\mu)$.*

**Proof.** By linearizing the system (1.1) at $EP_0$, the Jacobi matrix is given as

follows:

$$J(EP_0) = \begin{bmatrix} -\mu & 0 & 0 & \frac{-\beta_1\Lambda}{\mu} & \frac{-\beta_2\Lambda}{\mu} & \frac{-\beta_3\Lambda}{\mu} & \frac{-\beta_1\Lambda}{\mu} & 0 & r_1 & r_2 & r_3 \\ 0 & a_{22} & 0 & \frac{\beta_1\Lambda}{\mu} & 0 & \frac{p\beta_3\Lambda}{\mu} & \frac{\beta_1\Lambda}{\mu} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\tau_2-\mu & 0 & \frac{\beta_2\Lambda}{\mu} & a_{36} & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau_1 & 0 & -\sigma-\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau_2 & 0 & a_{55} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma\alpha & 0 & 0 & a_{77} & 0 & 0 & 0 & 0 \\ 0 & \delta\rho & 0 & \sigma(1-\alpha) & 0 & 0 & \rho & a_{88} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & \gamma_2 & a_{99} & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_3 & 0 & 0 & 0 & 0 & a_{00} & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_4 & 0 & 0 & 0 & 0 & -r_3-\mu \end{bmatrix}, \tag{3.3}$$

where $a_{22} = -\tau_1 - \rho\delta - \mu,\ a_{36} = \dfrac{(1-p)\beta_3\Lambda}{\mu},\ a_{55} = -\gamma_3 - d_2 - \mu,\ a_{66} = -d_3 - \gamma_4 - \mu,\ a_{77} = -\rho - \mu, a_{88} = -\gamma_2 - d_1 - \mu,\ a_{99} = -r_1 - \mu,\ a_{00} = -r_2 - \mu.$

The eigenvalues of the above Jacobi matrix are $-\mu$, $-r_1 - \mu$, $-r_2 - \mu$, $-r_3 - \mu$, $-\gamma_2 - d_1 - \mu$, $-d_3 - \gamma_4 - \mu$ and the roots of equation

$$x^2 + A_1 x + A_2 = 0, \tag{3.4}$$

and

$$x^3 + B_1 x^2 + B_2 x + B_3 = 0, \tag{3.5}$$

where

$$\begin{aligned} A_1 &= d_2 + \gamma_2 + 2\mu + \tau_2, \quad A_2 = (\tau_2 + \mu)(\gamma_3 + d_2 + \mu) - \beta_2\tau_2\Lambda/\mu, \\ B_1 &= 3\mu + \rho + \delta\rho + \sigma + \tau_1, \\ B_2 &= (\tau_1 + \delta\rho + \mu)(2\mu + \rho + \sigma) + (\sigma + \mu)(\mu + \rho) - \beta_1\Lambda\tau_1/\mu, \\ B_3 &= (\rho + \mu)(\tau_1 + \rho\delta + \mu)(\sigma + \mu) - \beta_1\tau_1\Lambda(\rho + \mu + \sigma\alpha)/\mu. \end{aligned}$$

We know that $A_1$ is positive. Further, $A_2 > 0$ if $\Re_2 < 1$. Then (3.4) satisfies the Routh Hurwitz criteria and it shows that roots of (3.4) are negative real.

Also, $B_1$ is positive. If $\beta_1\tau_1\Lambda < \mu(\tau_1 + \rho\delta + \mu)(\sigma + \mu)$ then $B_2$ & $B_3$ are positive and also $\Re_1 < 1$. Thus, $B_1B_2 > B_3$. Therefore, (3.5) satisfies the Routh Hurwitz criteria and it proves that the roots of (3.5) are negative real. This establish that the disease free equilibrium point is locally stable if $R_0 < 1$ and $\beta_1\tau_1\Lambda < \mu(\tau_1 + \rho\delta + \mu)(\sigma + \mu)$ . □

**Theorem 3.2.** *The equilibrium point $EP_1$ is local stable, if it satisfies the following conditions,*

$$\begin{aligned} A_i > 0,\ B_j > 0,\ (i = 1, 2, 3),\ (j = 1, 2, \cdots, 6), \\ A_1 A_2 > A_3,\ B_1 B_2 > B_3, B_3 M > B_1 N, \\ (B_3 N + B_6) M > B_5 M^2 + B_1 N^2, \\ (B_5 M - B_6) N (B_3 M - N B_1) > M(B_5 N - B_6) + (B_3 M - N B_1)^2 B_6 B_1, \end{aligned} \tag{3.6}$$

*where*

$$\begin{aligned} M &= B_1 B_2 - B_3, \\ N &= B_1 B_4 - B_5, \\ B_1 &= -a_{11} - a_{22} + \sigma + \mu - a_{77} - a_{88} - a_{99}, \\ B_2 &= -\beta_1 S^{01} \tau_1 - a_{22}(\sigma + \mu) + a_{22} a_{77} - (\sigma + \mu) a_{77} + a_{22} a_{88} - (\sigma + \mu) a_{88} + a_{77} a_{88} \\ &\quad + (a_{22} - (\sigma + \mu) + a_{77} + a_{88}) a_{99} + a_{11}(a_{22} - (\sigma + \mu) + a_{77} + a_{88} + a_{99}), \\ B_3 &= \beta_1 S^{01} a_{21} \tau_1 - \beta_1 S^{01} \tau_1 \sigma \alpha + \beta_1 S^{01} \tau_1 a_{77} + a_{22}(\sigma + \mu) a_{77} + \beta_1 S^{01} \tau_1 a_{88} \\ &\quad + a_{22}(\sigma + \mu) a_{88} - a_{22} a_{77} a_{88} + (\sigma + \mu) a_{77} a_{88} + (\beta_1 S^{01} \tau_1 + a_{22}(\sigma + \mu) \\ &\quad - a_{22} a_{77} + (\sigma + \mu) a_{77} - (a_{22} - (\sigma + \mu) + a_{77}) a_{88}) a_{99} + a_{11}(\beta_1 S^{01} \tau_1 - a_{77} a_{88} \\ &\quad + a_{22}((\sigma + \mu) - a_{77} - a_{88} - a_{99}) - (a_{77} + a_{88}) a_{99} + (\sigma + \mu)(a_{77} + a_{88} + a_{99})), \\ B_4 &= \beta_1 S^{01} a_{21} \tau_1 \sigma \alpha - \beta_1 S^{01} a_{21} \tau_1 a_{77} - \beta_1 S^{01} a_{21} \tau_1 a_{88} + \beta_1 S^{01} \tau_1 \sigma \alpha a_{88} \\ &\quad - \beta_1 S^{01} \tau_1 a_{77} a_{88} - a_{22}(\sigma + \mu) a_{77} a_{88} - r_1 a_{21} \delta \rho \gamma_2 \\ &\quad - (\beta_1 S^{01} a_{21} \tau_1 - \beta_1 S^{01} \tau_1 \sigma \alpha + \beta_1 S^{01} \tau_1 a_{77} + a_{22}(\sigma + \mu) a_{77} \\ &\quad + (\beta_1 S^{01} \tau_1 + a_{22}(\sigma + \mu) - a_{22} a_{77} + (\sigma + \mu) a_{77}) a_{88}) a_{99} \\ &\quad + a_{11}(\beta_1 S^{01} \tau_1 \sigma \alpha - a_{22}(\sigma + \mu) a_{77} - a_{22}(\sigma + \mu) a_{88} + a_{22} a_{77} a_{88} \\ &\quad - (\sigma + \mu) a_{77} a_{88} + (a_{77} a_{88} - (\sigma + \mu)(a_{77} + a_{88}) \\ &\quad + a_{22}(-(\sigma + \mu) + a_{77} + a_{88})) a_{99} - \beta_1 S^{01} \tau_1 (a_{77} + a_{88} + a_{99})), \\ B_5 &= a_{21} \tau_1 (\beta_1 S^{01} a_{77} a_{88} - r_1 \sigma \alpha \gamma_1) - r_1 a_{21}((\sigma + \mu) \delta \rho - a_{77} \delta \rho + \tau_1 \sigma (1 - \alpha)) \gamma_2 \\ &\quad + ((-\beta_1 S^{01} \tau_1 \sigma \alpha + \beta_1 S^{01} \tau_1 a_{77} + a_{22}(\sigma + \mu) a_{77}) a_{88} \\ &\quad + \beta_1 S^{01} a_{21} \tau_1 (a_{77} + a_{88})) a_{99} - \beta_1 S^{01} a_{21} \tau_1 \sigma \alpha (a_{88} + a_{99}) \\ &\quad + a_{11}((\beta_1 S^{01} \tau_1 + a_{22}(\sigma + \mu)) a_{77} a_{88} + ((\sigma + \mu) a_{77} a_{88} + \beta_1 S^{01} \tau_1 (a_{77} + a_{88}) \\ &\quad + a_{22}(-a_{77} a_{88} + (\sigma + \mu)(a_{77} + a_{88}))) a_{99} - \beta_1 S^{01} \tau_1 \sigma \alpha (a_{88} + a_{99})), \\ B_6 &= r_1 a_{21}((\sigma + \mu) a_{77} \delta \rho \gamma_2 + \tau_1 (\sigma \alpha a_{88} \gamma_1 + a_{77} \sigma (1 - \alpha) \gamma_2 - \sigma \alpha \rho \gamma_2)) \\ &\quad + ((a_{21} + a_{11}) \beta_1 S^{01} \tau_1 \sigma \alpha - (\beta_1 S^{01} \tau_1 (a_{21} + a_{11}) + a_{11} a_{22}(\sigma + \mu)) a_{77}) a_{88} a_{99}. \end{aligned} \tag{3.7}$$

$$\begin{aligned} A_1 &= -a_{33} - a_{55} - a_{66}, \\ A_2 &= -a_{35} \tau_2 + a_{33} a_{55} + a_{33} a_{66} + a_{55} a_{66}, \\ A_3 &= -a_{36} \tau_2 a_{65} + a_{35} \tau_2 a_{66} - a_{33} a_{55} a_{66}, \end{aligned} \tag{3.8}$$

*where $a_{ij}$ are defined later.*

**Proof.** Define the Jacobi matrix at the equilibrium point $EP_1$ of the model (1.1),

$$J(EP_1) = \begin{bmatrix} a_{11} & 0 & 0 & -\beta_1 S^{01} & -\beta_2 S^{01} & -\beta_3 S^{01} & -\beta_1 S^{01} & 0 & r_1 & r_2 & r_3 \\ a_{21} & a_{22} & 0 & \beta_1 S^{01} & 0 & p\beta_3 S^{01} & \beta_1 S^{01} & 0 & 0 & \beta_1 I_1^{01} & 0 \\ 0 & 0 & a_{33} & 0 & a_{35} & a_{36} & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau_1 & 0 & -\sigma-\mu & -\theta_1 b I_1^{01} & -b I_1^{01} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau_2 & 0 & a_{55} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma\alpha & -\theta_3 b A^{01} & -b A^{01} & a_{77} & 0 & 0 & 0 & 0 \\ 0 & \delta\rho & 0 & \sigma(1-\alpha) & 0 & 0 & \rho & a_{88} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta_2 R_1^{01} & -\beta_3 R_1^{01} & \gamma_1 & \gamma_2 & a_{99} & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_3 & 0 & 0 & 0 & 0 & a_{00} & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_4 & 0 & 0 & 0 & 0 & -r_3-\mu \end{bmatrix}, \tag{3.9}$$

where

$$\begin{aligned} a_{11} &= -\beta_1(I_1^{01}+A^{01})-\mu, & a_{22} &= -\tau_1-\rho\delta-\mu, \\ a_{35} &= \beta_2 S^{01}+\beta_2 R_1^{01}, & a_{36} &= (1-p)\beta_3 S^{01}+\beta_3 R_1^{01}, \\ a_{55} &= -\theta_2 b I_1^{01}-\gamma_3-d_2-\mu, & a_{65} &= \theta_1 b I_1^{01}+\theta_2 b I_1^{01}+\theta_3 b A^{01}, \\ a_{66} &= b(I_1^{01}+A^{01})-d_3-\gamma_4-\mu, & a_{77} &= -\rho-\gamma_1-\mu, \\ a_{88} &= -\gamma_2-d_1-\mu, & a_{99} &= -r_1-\mu, \\ a_{00} &= -r_2-\beta_1 I_1^{01}-\mu, & a_{21} &= \beta_1 I_1^{01}+A^{01}, \\ a_{33} &= -\tau_2-\mu. \end{aligned}$$

The Jacobian matrix $J(EP_1)$ has a characteristic polynomial of degree 11. We give the characteristic polynomial of $J(EP_1)$ in the following form:

$$(x-a_{00})(x+a_1)(x^3+A_1x^2+A_2x+A_3)(x^6+B_1x^5+B_2x^4+B_3x^3+B_4x^2+B_5x+B_6)=0,$$

where $a_1 = r_3+\mu$ and suppose $M, N, A_1, A_2, A_3, B_1, B_2, B_3, B_4, B_5, B_6$ are satisfies (3.6). Then according to the Routh-Hurwitz criteria, we can show that only negative eigenvalues exist. This proves that the model (1.1) is locally stable at the point $EP_1$. □

**Theorem 3.3.** *The equilibrium point $EP_2$ is local stable, if it satisfies the following conditions,*

$$\begin{aligned} A_i &> 0, & B_i &> 0, \\ A_1A_2 &> A_3, & B_1B_2 &> B_3, \\ A_3(A_1A_2-A_3) &> A_1A_4(A_1+1), & B_3(B_1B_2-B_3) &> B_1B_4(B_1+1), \end{aligned} \tag{3.10}$$

*with*

$$\begin{aligned} A_1 &= (-a_{00} - a_{11} + (\tau_2 + \mu) - a_{55}), \\ A_2 &= -a_{11}((\tau_2 + \mu) - a_{55}) + a_{00}(a_{11} - (\tau_2 + \mu) + a_{55}), \\ A_3 &= \beta_2 I_2^{02}(\tau_2 + \mu)a_{55} + a_{00}(a_{11}((\tau_2 + \mu) + a_{55})), \\ A_4 &= -\gamma_3 r_2 \beta_2 I_2^{02}\tau_2 - a_{00}(\beta_2 S^{02}\beta_2 I_2^{02}\tau_2), \end{aligned} \tag{3.11}$$

$$\begin{aligned} B_1 =& -(a_{22} + a_{44} + a_{66} + a_{77}), \\ B_2 =& -a_{24}\tau_1 + a_{44}a_{66} + a_{44}a_{77} + a_{66}a_{77} + a_{22}(a_{44} + a_{66} + a_{77}), \\ B_3 =& -a_{26}\tau_1 a_{64} + a_{24}\tau_1 a_{66} - a_{22}a_{44}a_{66} - a_{27}\tau_1\sigma\alpha \\ &+ (a_{24}\tau_1 - a_{22}a_{44} - (a_{22} + a_{44})a_{66})a_{77}, \\ B_4 =& a_{27}\tau_1 a_{66}\sigma\alpha - (a_{24}\tau_1 - a_{22}a_{44})a_{66}a_{77} - a_{26}\tau_1(\theta_3 b I_2^{02}\sigma\alpha - a_{64}a_{77}). \end{aligned} \tag{3.12}$$

*All $a_{ij}$'s are defined further.*

**Proof.** Let us define the Jacobin matrix at the Equilibrium point $EP_2$ of the model (1.1),

$$J(EP_2) = \begin{bmatrix} a_{11} & 0 & 0 & -\beta_1 S^{02} & -\beta_2 S^{02} & -\beta_3 S^{02} & -\beta_1 S^{02} & 0 & r_1 & r_2 & r_3 \\ 0 & a_{22} & 0 & a_{24} & 0 & a_{26} & a_{27} & 0 & 0 & 0 & 0 \\ \beta_2 I_2^{02} & 0 & a_{33} & 0 & \beta_2 S^{02} & a_{36} & 0 & 0 & \beta_2 I_2^{02} & 0 & 0 \\ 0 & \tau_1 & 0 & a_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau_2 & -\theta_2 b I_2^{02} & a_{55} & -b I_2^{02} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{64} & 0 & a_{66} & \theta_3 b I_2^{02} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma\alpha & 0 & 0 & a_{77} & 0 & 0 & 0 & 0 \\ 0 & \delta\rho & 0 & \sigma(1-\alpha) & 0 & 0 & \rho & a_{88} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & \gamma_2 & a_{99} & 0 & 0 \\ 0 & 0 & 0 & -\beta_1 R_2^{02} & \gamma_3 & -\beta_3 R_2^{02} & 0 & 0 & 0 & a_{00} & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_4 & 0 & 0 & 0 & 0 & -r_3 - \mu \end{bmatrix}, \tag{3.13}$$

where

$$\begin{aligned} &a_{11} = -\beta_2 I_2^{02} - \mu, && a_{22} = -\tau_1 - \rho\delta - \mu, && a_{24} = \beta_1(S^{02} + R_2^{02}), \\ &a_{66} = b I_2^{02} - d_3 - \gamma_4 - \mu, && a_{36} = (1-p)\beta_3 S^{02}, && a_{44} = -\theta_1 b I_2^{02} - \sigma - \mu, \\ &a_{55} = -\gamma_3 - d_2 - \mu, && a_{77} = -\rho - \gamma_1 - \theta_3 b I_2^{02} - \mu, && a_{26} = \beta_3(pS^{02} + R_2^{02}), \\ &a_{33} = -\tau_2 - \mu, && a_{88} = -\gamma_2 - d_1 - \mu, && a_{99} = -\gamma_2 - \beta_2 I_2^{02} - \mu, \\ &a_{00} = -r_2 - \mu, && a_{27} = \beta_1 S^{02} + \beta_3 R_2^{02}, && a_{64} = (\theta_1 + \theta_2) b I_2^{02}. \end{aligned}$$

The above Jacobian matrix has a characteristic polynomial of order 11. We can rewrite the $11^{th}$ order polynomial as $(x-a_{88})(x-a_{99})(x+r_3+\mu)(x^4+A_1x^3+A_2x^2+A_3x+A_4)(x^4+B_1x^3+B_2x^2+B_3x+B_4) = 0$.

Since the polynomial satisfies the inequalities defined (3.10) for the coefficient defined in (3.11). Then by Routh Hurwitz criteria says only negative eigenvalues exist, which shows that the model (1.1) is locally stable at the point $EP_2$. □

Next, let us define the Routh-Hurwitz criteria for the $11^{th}$ degree polynomial. Consider the polynomial as follows:

$$x^{11}+a_1x^{10}+a_2x^9+a_3x^8+a_4x^7+a_5x^6+a_6x^5+a_7x^4+a_8x^3+a_9x^2+a_{10}x+a_{11}=0. \tag{3.14}$$

Here $a_i's$ are real for $i=1,2,\cdots,11$. Then the Routh Hurwitz criteria states that if

$$\begin{aligned}
&a_i>0 \ \ (i=1,2,\cdots,11),\\
&b_1=a_2-a_3/a_1>0,\ b_i=a_{2*i}-a_{2*i+1}/a_1, \quad (i=2,3,4,5), \qquad b_i=0\ (i>5),\\
&c_1=a_3-b_2a_2/b_1>0,\ c_i=a_{2*i+1}-b_{i+1}a_1/b_1, \quad (i=2,3,4,5),\ c_i=0\ (i>5),\\
&d_1=b_2-c_2b_1/c_1>0,\ d_i=b_{i+1}-c_{i+1}b_1/c_1, \quad (i=2,3,4), \qquad d_i=0\ (i>4),\\
&e_1=c_2-d_2c_1/d_1>0,\ e_i=c_{i+1}-d_{i+1}c_1/d_1, \quad (i=2,3,4), \qquad e_i=0\ (i>4),\\
&f_1=d_2-e_2d_1/e_1>0,\ f_i=d_{i+1}-e_{i+1}d_1/e_1, \quad (i=2,3), \qquad f_i=0\ (i>3),\\
&g_1=e_2-f_2e_1/f_1>0,\ g_i=e_{i+1}-f_{i+1}e_1/f_1, \quad (i=2,3), \qquad g_i=0\ (i>3),\\
&h_1=f_2-g_2f_1/g_1>0,\ h_2=f_3-g_3f_1/g_1, \qquad h_i=0\ (i>2),\\
&k_1=g_2-h_2g_1/h_1>0,\ k_2=g_3-h_3g_1/h_1, \qquad k_i=0\ (i>2),\\
&l_1=h_2-k_2h_1/k_1>0, \qquad l_i=0\ (i>1),
\end{aligned} \tag{3.15}$$

then the equation (3.2) have only real negative roots.

**Theorem 3.4.** *Suppose that $R_0>1$and satisfies the Routh-Hurwitz criteria* (3.15)*, then the endimic equilibrium point $EE$ is local stable.*

**Proof.** Let us define the Jacobi matrix at the Equilibrium point $EE$ for the model (1.1),

$$J(EE)=\begin{bmatrix}
a_{11} & 0 & 0 & -\beta_1S^* & -\beta_2S^* & -\beta_3S^* & -\beta_1S^* & 0 & r_1 & r_2 & r_3\\
a_{21} & a_{22} & 0 & a_{24} & 0 & a_{26} & \beta_1S^* & 0 & 0 & a_{20} & 0\\
a_{31} & 0 & -\tau_2-\mu & 0 & a_{35} & a_{36} & 0 & 0 & a_{39} & 0 & 0\\
0 & \tau_1 & 0 & a_{44} & -\theta_1bI_1^* & -bI_1^* & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \tau_2 & -\theta_2bI_2^* & a_{55} & -bI_2^* & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & a_{64} & a_{65} & a_{66} & a_{67} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \sigma\alpha & -\theta_3bA^* & -bA^* & a_{77} & 0 & 0 & 0 & 0\\
0 & \delta\rho & 0 & \sigma(1-\alpha) & 0 & 0 & \rho & a_{88} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -\beta_2R_1^* & -\beta_3R_1^* & \gamma_1 & \gamma_2 & a_{99} & 0 & 0\\
0 & 0 & 0 & -\beta_1R_2^* & \gamma_3 & -\beta_3R_2^* & 0 & 0 & 0 & a_{00} & 0\\
0 & 0 & 0 & 0 & 0 & \gamma_4 & 0 & 0 & 0 & 0 & -r_3-\mu
\end{bmatrix}. \tag{3.16}$$

Here

$$
\begin{aligned}
a_{21} &= \beta_1(I_1^* + A^*) + p\beta_3 I_3^*, & a_{22} &= -\tau_1 - \rho\delta - \mu,\\
a_{11} &= -\beta_1(I_1^* + A^*) + \beta_2 I_2^* - \beta_3 I_3^* - \mu, & a_{24} &= \beta_1 S^* + \beta_1 R_2^*,\\
a_{26} &= p\beta_3 S^* + \beta_3 R_2^*, & a_{65} &= \theta_1 b I_1^* + \theta_2 b I_1^* + bI_3 +^* \theta_3 bA^*,\\
a_{31} &= \beta_2 I_2^* + (1-p)\beta_3 I_3^*, & a_{35} &= \beta_2 S^* + \beta_2 R_1^*,\\
a_{36} &= (1-p)\beta_3 S^* + \beta_3 R_1^*, & a_{39} &= \beta_2 I_2^* + \beta_3 I3^*,\\
a_{44} &= -\theta_1 b I_2^* - bI_3^* - \sigma - \mu, & a_{55} &= -\theta_2 b I_1^* - bI_3^* - \gamma_3 - d_2 - \mu,\\
a_{64} &= \theta_1 b I_2^* + bI_3^* + \theta_2 b I_2^*, & a_{20} &= \beta_1 I_1^* + \beta_3 I_3^*,\\
a_{66} &= b(I_1^* + I_2^* + A^*) - d_3 - \gamma_4 - \mu, & a_{67} &= \theta_3 b I_2^* + bI_3^*,\\
a_{88} &= -\gamma_2 - d_1 - \mu, & a_{77} &= -\rho - \gamma_1 - \theta_3 b I_2^* - bI_3^* - \mu,\\
a_{00} &= -r_2 - \mu, & a_{99} &= -\gamma_2 - \beta_2 I_2^* - \beta_3 I_3^* - \mu,\\
a_{20} &= \beta_1 I_1 + \beta_3 I_3.
\end{aligned}
$$

The above Jacobian matrix has a characteristic polynomial of order 11, if that polynomial satisfies the Routh-Hurwitz criteria (3.15), then the model (1.1) is locally stable at the equilibrium point $EE$. □

Next, we prove the global stability of the disease-free equilibrium point of the model (1.1) using the Lyapunov function.

**Theorem 3.5.** *The disease free equilibrium point of the model* (1.1) *is globally stable if the model parameter satisfies* $\mu^2 > \max\{\beta_1\Lambda, \beta_2\Lambda, \beta_3\Lambda\}$.

**Proof.** We define the Lyapunov function as,

$$L = \left(S - S^* - S^* ln\left(\frac{S}{S^*}\right)\right) + E_1 + E_2 + I_1 + I_2 + I_3 + A + Q + R_1 + R_2 + R_3. \quad (3.17)$$

Differentiate (3.17), we get

$$
\begin{aligned}
dL &\le \left(1 - \frac{S^*}{S}\right) dS + dE_1 + dE_2 + dI_1 + dI_2 + dI_3 + dA + dQ + dR_1 + dR_2 + dR_3\\
&\le -d_1 Q + d_2 I_2 + d_3 I_3 - \mu N - \frac{\Lambda^2}{\mu S} + (\beta_1 (I_1 + A) + \beta_2 I_2 + \beta_3 I_3)\frac{\Lambda}{\mu}\\
&\le \left(\frac{\beta_1\Lambda}{\mu} - \mu\right) I_1 + \left(\frac{\beta_2\Lambda}{\mu} - \mu\right) I_2 + \left(\frac{\beta_3\Lambda}{\mu} - \mu\right) I_3,
\end{aligned}
\quad (3.18)
$$

provided $\mu^2 > \max\{\beta_1\Lambda, \beta_2\Lambda, \beta_3\Lambda\}$. Then $dL < 0$ and it shows that the disease free equilibrium point is globally stable. □

## 4. Sensitivity analysis

In this section, we study the sensitivity index of the basic reproduction number with respect to the parameter values defined in the model (1.1).

First, we define sensitivity as in [6]. The sensitivity of $Y$ with respect to the parameter $r$ is defined as

$$K = \frac{r}{Y}\frac{dY}{dr}.$$

Suppose that $K > 0$, then function $Y$ is propositional to $r$. If $K < 0$, then the function $Y$ is inversely propositional to $r$.

Now, we perform for the basic reproduction number $R_0$. We know that $R_0$ is equal to the maximum of $\mathfrak{R}_1$ and $\mathfrak{R}_2$. From (3.2), it is clear that $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are functions of $\Lambda$ and $\mu$ with other parameters. Therefore, first, we study the sensitivity index of $R_0$ with respect to $\Lambda$ and $\mu$.

$$\frac{\Lambda}{\mathfrak{R}_i}\frac{d\mathfrak{R}_i}{d\Lambda} = 1 > 0 \quad (i = 1, 2). \tag{4.1}$$

By (4.1), for $i = 1, 2$, it is clear that $\dfrac{\Lambda}{\mathfrak{R}_i}\dfrac{d\mathfrak{R}_i}{d\Lambda}$ are positive and equal to one. This indicates that $R_0$ increases/decreases when $\Lambda$ increases/decreases. Next, we study with respect to the parameter $\mu$.

$$\begin{aligned}
\frac{\mu}{\mathfrak{R}_1}\frac{d\mathfrak{R}_1}{d\mu} &= \frac{-\mu}{\rho+\gamma_1+\mu+\sigma\alpha}\left(\frac{\sigma\alpha}{\rho+\gamma_1+\mu} + \left(\frac{1}{\mu} + \frac{1}{(\sigma+\mu)} + \frac{1}{\tau_1+\rho\delta+\mu}\right)\right) &< 0,\\
\frac{\mu}{\mathfrak{R}_2}\frac{d\mathfrak{R}_2}{d\mu} &= -\mu\left(\frac{1}{\mu} + \frac{1}{(\tau_2+\mu)} + \frac{1}{\gamma_3+d_2+\mu}\right) &< 0.
\end{aligned} \tag{4.2}$$

In view of (4.2), if $\mu$ increases/decreases then $R_0$ decreases/increases. Similarly, we study for $\mathfrak{R}_1$ with respect to the all other model parameters as below.

$$\begin{aligned}
\frac{\beta_1}{\mathfrak{R}_1}\frac{d\mathfrak{R}_1}{d\beta_1} &= 1 &> 0,\\
\frac{\tau_1}{\mathfrak{R}_1}\frac{d\mathfrak{R}_1}{d\tau_1} &= \frac{\rho\delta+\mu}{\tau_1+\rho\delta+\mu} &> 0,\\
\frac{\alpha}{\mathfrak{R}_1}\frac{d\mathfrak{R}_1}{d\alpha} &= \frac{\sigma\alpha}{(\rho+\gamma_1+\mu+\sigma\alpha)} &> 0.
\end{aligned} \tag{4.3}$$

(4.3) yields that if $\beta_1, \tau_1$ and $\alpha$ are increasing then $R_0$ is increasing otherwise it decreases.

$$\begin{aligned}
\frac{\rho}{\mathfrak{R}_1}\frac{d\mathfrak{R}_1}{d\rho} &= \frac{-\rho}{(\rho+\gamma_1+\mu+\sigma\alpha)}\left(\frac{\sigma\alpha}{\rho+\gamma_1+\mu} + \frac{(\rho+\gamma_1+\mu+\sigma\alpha)\delta}{\tau_1+\rho\delta+\mu}\right) &< 0,\\
\frac{\gamma_1}{\mathfrak{R}_1}\frac{d\mathfrak{R}_1}{d\gamma_1} &= \frac{-\gamma_1\sigma\alpha}{(\rho+\gamma_1+\mu+\sigma\alpha)(\rho+\gamma_1+\mu)} &< 0,\\
\frac{\delta}{\mathfrak{R}_1}\frac{d\mathfrak{R}_1}{d\delta} &= -\frac{\delta\rho}{(\tau_1+\rho\delta+\mu)} &< 0.
\end{aligned} \tag{4.4}$$

From (4.4), if $\rho, \gamma_1$ and $\delta$ are increasing then $R_0$ is decreasing otherwise it increases.

$$\frac{\sigma}{\mathfrak{R}_1}\frac{d\mathfrak{R}_1}{d\sigma} = \frac{\sigma}{\rho+\gamma_1+\mu\sigma\alpha}\left(\frac{\alpha\sigma-\rho-\gamma_1-\mu}{\sigma+\mu}\right). \tag{4.5}$$

We know that all the model parameters are positive and from (4.5), suppose $\alpha\sigma < \rho+\gamma_1+\mu$ then $R_0$ is inversely propositional to $\sigma$. Otherwise, $R_0$ is propositional to $\sigma$.

As similar as $\mathfrak{R}_1$, we perform analysis for $\mathfrak{R}_2$ with respect to the all other model parameters.

$$\begin{aligned} \frac{\gamma_3}{\mathfrak{R}_2}\frac{d\mathfrak{R}_2}{d\gamma_3} &= \frac{-\gamma_3}{\gamma_3+d_2+\mu} < 0, \\ \frac{d_2}{\mathfrak{R}_2}\frac{d\mathfrak{R}_2}{dd_2} &= \frac{-d_2}{\gamma_3+d_2+\mu} < 0. \end{aligned} \tag{4.6}$$

By (4.6), if $\gamma_3$ ana $d_2$ are increasing then $R_0$ is decreasing otherwise it increases.

$$\begin{aligned} \frac{\beta_2}{\mathfrak{R}_2}\frac{d\mathfrak{R}_2}{d\beta_2} &= 1 \qquad\quad > 0, \\ \frac{\tau_2}{\mathfrak{R}_2}\frac{d\mathfrak{R}_2}{d\tau_2} &= \frac{\mu}{\tau_2+\mu} > 0. \end{aligned} \tag{4.7}$$

In view of (4.7), if $\beta_2$, and $\tau_2$ are increasing then $R_0$ is increasing otherwise it decreases.

The sensitivity index of the model parameters are numerically shown below. Further, all the parameter values $\Gamma, \beta_1,\ \beta_2,\ \beta_3,\ \tau_1,\ \tau_2,\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \theta_1,\ \theta_2,\ \theta_3,\ r_1,\ r_2,\ r_3,\ \mu,\ \rho,\ \delta,\ \alpha,\ \sigma,\ p,\ b,\ d_1,\ d_2,\ d_3$ of the model are assumed as in the Table 2.

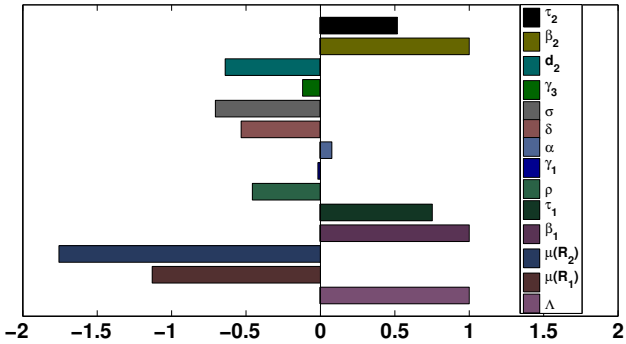


**Figure 2.** Plot represents the sensitivity of the parameters in the basic reproduction number of co-infection model.

# 5. Optimal control problem

In this section, we introduce the control parameters $u_1(t), u_2(t)$, and $u_3(t)$ in recovery classes of both diseases for the proposed co-infection model, and the same is given below in (5.1). Then, we study the existence of optimal control for the newly constructed co-infection control problem. Further, the first-order optimality conditions are derived from attaining the optimal solution of the co-infection model (5.1).

Control variables are introduced in the proposed model in order to reduce the reinfection of the disease. We transform, $r_i\ i = 1, 2, 3$ the rate of person going to the class $S$ from $R_i\ i = 1, 2, 3$ as control variables $u_i(t),\ i = 1, 2, 3$ depends on time

**Table 2.** Parameter values of the co-infection model (1.1)

| $\Lambda$ | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\tau_1$ | $\tau_2$ | $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | $\gamma_4$ | $\theta_1$ | $\theta_2$ | $\theta_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.16 | 0.135 | 0.23 | 0.34 | 0.28 | 0.3 | 0.53 | 0.15 | 0.5 | 0.3 | 0.28 | 0.42 |

| $r_1$ | $r_2$ | $r_3$ | $\mu$ | $\rho$ | $\delta$ | $\alpha$ | $\sigma$ | $p$ | $b$ | $d_1$ | $d_2$ | $d_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.9 | 0.5 | 0.87 | 0.3 | 0.73 | 1 | 0.23 | 0.48 | 0.67 | 0.41 | 0.1 | 0.8 | 0.1 |

$t$. Further, the remaining parameters are unchanged as in (1.1), and the resulting optimal control problem is given as follows:

$$\left.\begin{aligned}
\frac{dS}{dt} &= \Lambda - \beta_1 S\left(I_1 + A\right) - \beta_3 S I_3 - \beta_2 S I_2 + u_1(t)R_1 + u_2(t)R_2 + u_3(t)R_3 - \mu S,\\
\frac{dE_1}{dt} &= \beta_1 S\left(I_1 + A\right) + p\beta_3 S I_3 - \tau_1 E_1 - \rho\delta E_1 + \beta_1 I_1 R_2 + \beta_3 I_3 R_2 - \mu E_1,\\
\frac{dE_2}{dt} &= \beta_2 S I_2 + (1-p)\beta_3 S I_3 - \tau_2 E_2 + \beta_2 I_2 R_1 + \beta_3 I_3 R_1 - \mu E_2,\\
\frac{dI_1}{dt} &= \tau_1 E_1 - \theta_1 b I_1 I_2 - b I_1 I_3 - \sigma I_1 - \mu I_1,\\
\frac{dI_2}{dt} &= \tau_2 E_2 - \theta_2 b I_1 I_2 - b I_2 I_3 - \gamma_3 I_2 - d_2 I_2 - \mu I_2,\\
\frac{dI_3}{dt} &= \theta_1 b I_1 I_2 + b I_1 I_3 + \theta_2 b I_1 I_2 + b I_2 I_3 + \theta_3 b A I_2 + b A I_3 - d_3 I_3 - \gamma_4 I_3 - \mu I_3,\\
\frac{dA}{dt} &= \sigma\alpha I_1 - \rho A - \gamma_1 A - \theta_3 b A I_2 - b A I_3 - \mu A,\\
\frac{dQ}{dt} &= \sigma(1-\alpha) I_1 + \rho A - \gamma_2 Q - d_1 Q + \delta\rho E_1 - \mu Q,\\
\frac{dR_1}{dt} &= \gamma_1 A + \gamma_2 Q - u_1(t)R_1 - \beta_2 I_2 R_1 - \beta_3 I_3 R_1 - \mu R_1,\\
\frac{dR_2}{dt} &= \gamma_3 I_2 - u_2(t)R_2 - \beta_1 I_1 R_2 - \beta_3 I_3 R_2 - \mu R_2,\\
\frac{dR_3}{dt} &= \gamma_4 I_3 - u_3(t)R_3 - \mu R_3.
\end{aligned}\right\}. \tag{5.1}$$

We need to minimize the following objective functional for the proposed optimal control problem (5.1).

$$J(u_1, u_2, u_3) = \int_0^T \left(\epsilon_1 R_1 + \epsilon_2 R_2 + \epsilon_3 R_3 + \sum_{i=1}^{3} \frac{\rho_i u_i^2}{2}\right) dt. \tag{5.2}$$

Here $\rho_i$ describe the cost coefficient and the terms $\frac{\rho_i u_i^2}{2}$ represents cost for increase the immunity of the persons. Also, the terms $\epsilon_i$ are the coefficients of balancing factors. The set of control functions are defined as

$$U = \{(u_1, u_2, u_3) | u_i(t) \in L^2[0,T] : 0 \leq u_i(t) \leq u_{imax}, i = 1,2,3\}. \tag{5.3}$$

The objective is to determine the optimal control value $u_1^*, u_2^*$ and $u_3^*$ that satisfies

$$J(u_1^*, u_2^*, u_3^*) = \min_U J(u_1, u_2, u_3). \tag{5.4}$$

**Theorem 5.1.** *There exists an optimal control functions $u^*(t) = (u_1^*, u_2^*, u_3^*)$ and the corresponding solution trajectories to the initial value problem* (5.1) *such that it satisfies* (5.4)

**Proof.** To show the existence of the control variables, the objective functional satisfies the following properties:

(i) set of control variables and the state variables is non-empty. It is valid because the nonlinear functions of (5.1) are uniformly Lipschitz continuous.

(ii) the space of control is closed and convex. We know that $L^p$ is closed and convex, and this proves the result.

(iii) the RHS of the model (5.1) is bounded. The boundedness of the state and control variables prove the result.

(iv) the objective functional is convex with respect to control variables $u_i$ $(i = 1,2,3)$. Let $\tilde{u},\ \tilde{v} \in U$ and $0 < \varsigma < 1$. From (5.2), we get

$$\begin{aligned} J(\varsigma\tilde{u} + (1-\varsigma)\tilde{v}) &\leq \int_0^T \left( \epsilon_1 R_1 + \epsilon_2 R_2 + \epsilon_3 R_3 + \frac{1}{2}\sum_{i=1}^{3} \varsigma\rho_i \tilde{u}_i^2 + (1-\varsigma)\rho_i \tilde{v}_i^2 \right) dt \\ &\leq \varsigma J(\tilde{u}) + (1-\varsigma) J(\tilde{v}). \end{aligned} \tag{5.5}$$

(v) there exists a positive constant $C_1,\ C_2$ such that:

$$\begin{aligned} \mathbb{L}(X,t,u) &= \epsilon_1 R_1 + \epsilon_2 R_2 + \epsilon_3 R_3 + \sum_{i=1}^{3} \frac{\rho_i u_i^2}{2} \\ &\geq \frac{\rho_1 u_1^2}{2} + \frac{\rho_2 u_2^2}{2} + \frac{\rho_3 u_3^2}{2} \\ &\geq \min\left\{\frac{\rho_1}{2}, \frac{\rho_2}{2}, \frac{\rho_3}{2}\right\}(u_1^2 + u_2^2 + u_3^2) \\ &\geq C_1|u|^2 - C_2. \end{aligned} \tag{5.6}$$

Here $C_1 = \min\left\{\frac{\rho_1}{2}, \frac{\rho_2}{2}, \frac{\rho_3}{2}\right\}$, $C_2 > 0$. This shows the bound of Lagrangian function $\mathbb{L}$ of the objective function $J$. □

**Theorem 5.2.** *Let $u^*(t)$ is the optimal control variable set for the optimal control model* (1.1) *and $X^*(t)$ is the corresponding optimal solution to the model. Then*

*there exists a co-state variable* $\lambda_i$ $(i = 1, \cdots, 11)$ *which satisfies the following:*

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= \lambda_1(-\beta_1(I_1+A) - p\beta_3 I_3 - \beta_2 I_2 - (1-p)\beta_3 I_3 - \mu) + \lambda_2(\beta_1(I_1+A) + p\beta_3 I_3) \\
&\quad + \lambda_3(\beta_2 I_2 + (1-p)\beta_3 I_3), \\
\frac{d\lambda_2}{dt} &= \lambda_2(-\tau_1 - \rho\delta - \mu) + \lambda_4(\tau_1) + \lambda_8(\rho\delta), \\
\frac{d\lambda_3}{dt} &= \lambda_3(-\tau_2 - \mu) + \lambda_4(\tau_2), \\
\frac{d\lambda_4}{dt} &= \lambda_1(-\beta_1 S) + \lambda_2(\beta_1 S + \beta_1 R_2) + \lambda_4(-\theta_1 b I_2 - b I_3 - \sigma - \mu) + \lambda_5(-\theta_2 b I_2) \\
&\quad + \lambda_6(\theta_1 b I_2 + b I_3 + \theta_2 b I_2) + \lambda_7(\sigma\alpha) + \lambda_8(\sigma(1-\alpha)) + \lambda_{10}(-\beta_1 R_2), \\
\frac{d\lambda_5}{dt} &= \lambda_1(-\beta_2 S) + \lambda_3(\beta_2 S + \beta_2 R_1) + \lambda_4(-\theta_1 b I_1) + \lambda_5(-\theta_2 b I_1 - b I_3 - \gamma_3 - d_2 - \mu) \\
&\quad + \lambda_6(\theta_1 b I_1 + \theta_2 b I_1 + b I_3 + \theta_3 b A) - \lambda_7\theta_3 b A - \lambda_9\beta_2 R_1 + \lambda_{10}\gamma_3, \\
\frac{d\lambda_6}{dt} &= \lambda_1(-\beta_3 S) + \lambda_2(\beta_3(pS + R_2)) + \lambda_3(\beta_3((1-p)S + R_1)) - \lambda_4 b I_1 - \lambda_5 b I_2 \\
&\quad + \lambda_6(b(I_1 + I_2 + A) - d_3 - \gamma_4 - \mu) - \lambda_7 b A - \lambda_9\beta_3 R_1 - \lambda_{10}(\beta_3 R_2) + \lambda_{11}\gamma_4, \\
\frac{d\lambda_7}{dt} &= \lambda_1(-\beta_1 S) + \lambda_2(\beta_2 S) + \lambda_6(\theta_3 b I_2 + b I_3) + \lambda_7(-\rho - \gamma_1 - \theta_3 b I_2 - b I_3 - \mu) \\
&\quad + \lambda_8\rho + \lambda_9\gamma_1, \\
\frac{d\lambda_8}{dt} &= \lambda_8(-\gamma_2 - d_1 - \mu) + \lambda_9\gamma_2, \\
\frac{d\lambda_9}{dt} &= \lambda_1(u_1) + \lambda_3(\beta_2 I_2 + \beta_3 I_3) + \lambda_9(-u_1 - \beta_2 I_2 - \beta_3 I_3 - \mu), \\
\frac{d\lambda_{10}}{dt} &= \lambda_1(u_2) + \lambda_2(\beta_1 I_1 + \beta_3 I_3) + \lambda_{10}(-u_2 - \beta_1 I_1 - \beta_3 I_3 - \mu), \\
\frac{d\lambda_{11}}{dt} &= \lambda_1(u_3) + \lambda_{11}(-u_3 - \mu) + \epsilon_3,
\end{aligned} \tag{5.7}$$

*with the boundary conditions* $\lambda_i(T) = 0$ $(i = 1, \cdots, 11)$. *Further,*

$$\begin{aligned}
u_1^* &= \min\left\{\max\left\{u_{min}, \frac{(\lambda_9 - \lambda_9)R_1}{\rho_1}\right\}, u_{max}\right\}, \\
u_2^* &= \min\left\{\max\left\{u_{min}, \frac{(\lambda_{10} - \lambda_1)R_2}{\rho_2}\right\}, u_{max}\right\}, \\
u_3^* &= \min\left\{\max\left\{u_{min}, \frac{(\lambda_{11} - \lambda_1)R_3}{\rho_3}\right\}, u_{max}\right\}.
\end{aligned} \tag{5.8}$$

**Proof.** Using the Pontryagin's maximum principle, we transform the objective

functional $J$ to minimize the Hamiltonian function $H$ and it is defined as follows:

$$\begin{aligned}
H &= \mathbb{L} + \sum_{i=1}^{11} \lambda_i F_i(X) \\
&= \epsilon_1 R_1 + \epsilon_2 R_2 + \epsilon_3 R_3 + \sum_{i=1}^{3} \frac{\rho_i u_i^2}{2} + \lambda_1(\Lambda - \beta_1 S\,(I_1 + A) - p\beta_3 S I_3 - \beta_2 S I_2 \\
&\quad -(1-p)\beta_3 S I_3 + r_1 R_1 + r_2 R_2 + r_3 R_3 - \mu S) + \lambda_2(\beta_1 S\,(I_1 + A) + p\beta_3 S I_3 \\
&\quad -\tau_1 E_1 - \rho\delta E_1 + \beta_1 I_1 R_2 + \beta_3 I_3 R_2 \\
&\quad -\mu E_1) + \lambda_3(\beta_2 S I_2 + (1-p)\beta_3 S I_3 - \tau_2 E_2 + \beta_2 I_2 R_1 + \beta_3 I_3 R_1 - \mu E_2) \\
&\quad +\lambda_4(\tau_1 E_1 - \theta_1 b I_1 I_2 - b I_1 I_3 - \sigma I_1 - \mu I_1) \\
&\quad +\lambda_5(\tau_2 E_2 - \theta_2 b I_1 I_2 - b I_2 I_3 - \gamma_3 I_2 - d_2 I_2 - \mu I_2) + \lambda_6(\theta_1 b I_1 I_2 + b I_1 I_3 \\
&\quad +b(I_2(\theta_2 I_1 + I_3 + \theta_3 A) + (A I_3) - d_3 I_3 - \gamma_4 I_3 - \mu I_3) \\
&\quad +\lambda_7(\sigma\alpha I_1 - \rho A - \gamma_1 A - (\theta_3 I_2 + I_3) b A - \mu A) + \lambda_8(\sigma(1-\alpha) I_1 + \rho A \\
&\quad -\gamma_2 Q - d_1 Q + \delta\rho E_1 - \mu Q) + \lambda_9(\gamma_1 A + \gamma_2 Q - r_1 R_1 - \beta_2 I_2 R_1 \\
&\quad -\beta_3 I_3 R_1 - \mu R_1) + \lambda_{10}(\gamma_3 I_2 - r_2 R_2 - \beta_1 I_1 R_2 - \beta_3 I_3 R_2 - \mu R_2) \\
&\quad +\lambda_1 1(\gamma_4 I_3 - r_3 R_3 - \mu R_3).
\end{aligned} \tag{5.9}$$

Differentiate (5.9) with respect to the state variables $X(t)$. Then, we get the co-state equations (5.7). Further, differentiate (5.9) with respect to the control variables $u(t)$, we get

$$\begin{aligned}
\frac{\partial H}{\partial u_1} &= \rho_1 u_1 + (\lambda_1 - \lambda_9) R_1 = 0, \\
\frac{\partial H}{\partial u_2} &= \rho_2 u_2 + (\lambda_1 - \lambda_{10}) R_2 = 0, \\
\frac{\partial H}{\partial u_3} &= \rho_3 u_3 + (\lambda_1 - \lambda_{11}) R_3 = 0.
\end{aligned} \tag{5.10}$$

From the above, it is easy to derive the optimality conditions (5.8). □

# 6. Numerical simulations

In this section, we show some computations of the proposed co-infection COVID-19 and HIV mathematical model. First, we determine the changes in basic reproduction number $R_0$ with respect to model parameters. Then, the evolution of the unknown variables of the model equations is analyzed with disease-free equilibrium and endemic equilibrium points. Finally, the co-infection model with optimal control is solved numerically, and their results are presented.

For all computations initial values of the unknown variables in the model are assumed as $X(0) = (10, 5, 5, 3, 3, 2, 1, 2, 1, 1, 1)$. Further, all the model parameters are assumed as in Table 2. The basic reproduction number $R_0$ is calculated for the Table 2 values and it is given as $\max\{0.4015,\ 0.869\}$. Now, we analyze the change in $R_0$ in the following four cases

Case (i): increase in recruitment rate ($\Lambda$),

Case (ii): increase in infection rate ($\beta_1$),

Case (iii): increase in testing rate ($\rho$),

Case (iv): increase in recovery rate of HIV individuals ($\gamma_3$),

and the results are depicted in Fig. 3 (a)-(d). In all the above four cases, all other parameters are fixed as in Table 2 except the parameter in the corresponding case. In case (i), an increase in the recruitment rate of the susceptible class increases the $R_0$ value. For large $\Lambda$, $R_0$ is greater than one, and it shows that the disease is endemic, see Fig. 3(a) . In case (ii), we observe that $R_0$ is constant till $\beta_1$ is less than 0.3. Then, $R_0$ is increasing and infection becomes endemic for large $\beta_1$, see Fig. 3(b). In case (iii), we observe that $R_0$ is decreasing till $\rho$ is less than 0.1. Then, $R_0$ becomes constant and infection reduces for large $\rho$ ($R_0 < 1$) as shown in Fig. 3(c). In case (iv), recovery rate of $HIV$ population increases then it decrease the value of $R_0$, see Fig. 3(d).

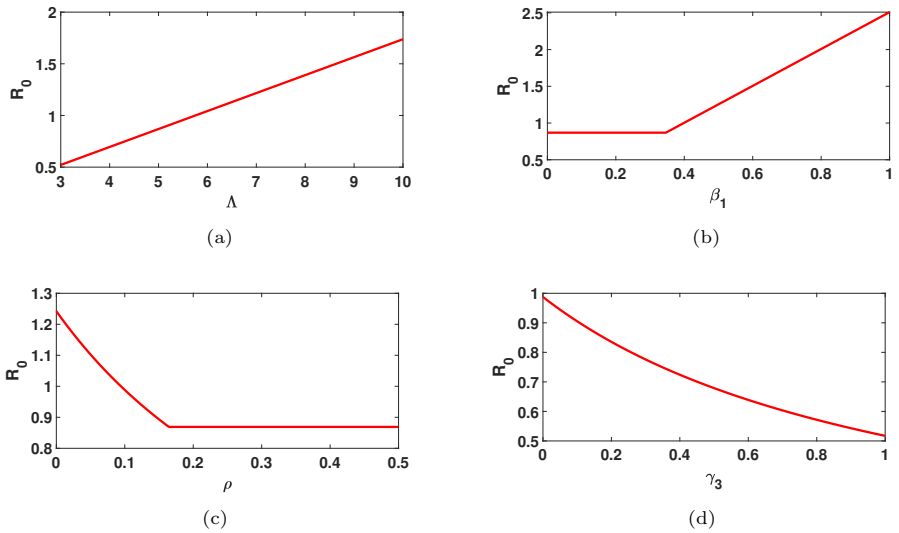


**Figure 3.** Plots represent the change in the value of $R_0$ with respect to the parameter values. In (a) $R_0$ with respect to $\Lambda$. (b) $R_0$ with respect to $\beta_1$. (c) $R_0$ with respect to $\rho$. (d) $R_0$ with respect to $\gamma_3$. Except the parameter used in the x-axis all other values are taken from the Table. 2.

First, we consider the model parameters as in Table 2. This parameter values satisfies the hypothesis of Theorem 3.1, that is, $\beta_1\tau_1\Lambda < \mu(\tau_1 + \rho\delta + \mu)(\sigma + \mu)$. In this case, the basic reproduction number, $R_0$ is less than 1. Computational results are depicted in Fig. 4 as dotted lines. In the next case, we assume that values of $\beta_i$ ($i = 1, 2, 3$) are twice as in the Table 2, and the remaining parameters are not changed in Table. 2. Therefore, it is easy to find that the basic reproduction number, $R_0$ is greater than 1. Numerical results are shown in Fig. 4 as solid lines. When $R_0 < 1$, then the model converges to a disease-free equilibrium point. However, when $R_0 > 1$, the model converges to an endemic equilibrium point. It is clearly visible in Fig. 4(a)-4(e).

Finally, we investigate the effects of control variables in the co-infection model equations. Admissible control values are assumed as lies between 0 to 0.9. Other control parameter values are taken as in Table. 3. As we have seen earlier, control parameters $u_i(t)$ are assumed in the reinfection rates of $R_i$ ($i = 1, 2, 3$) respectively. Numerical simulations are performed for model equations with and without con-

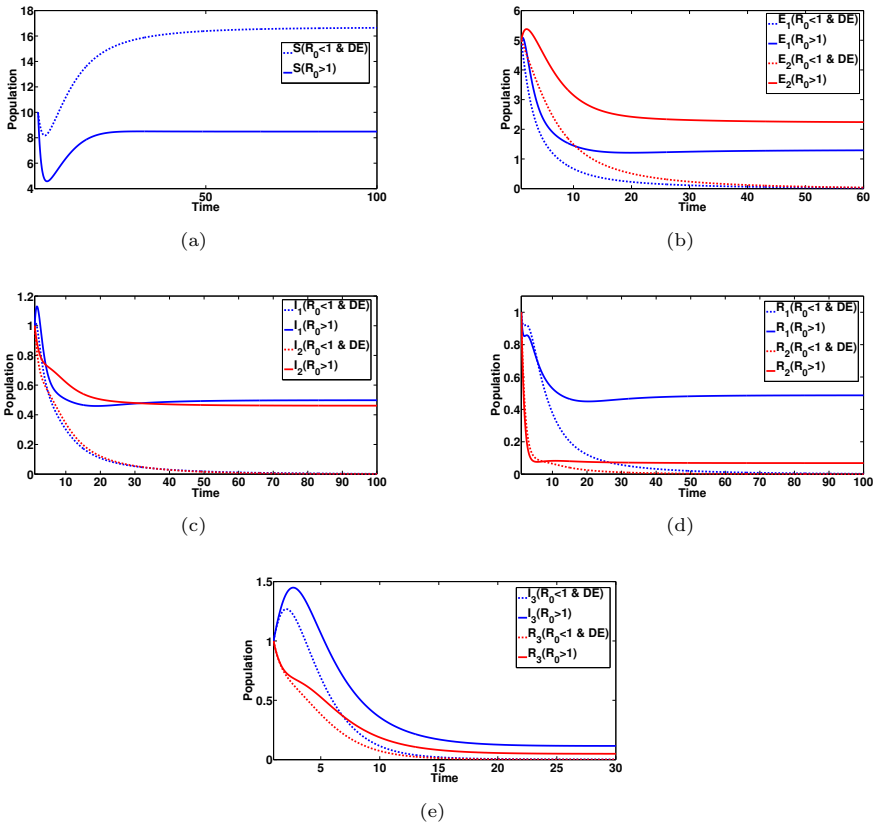


**Figure 4.** Plot (a) represents the evolution of $S$ of the model (1.1). Plot (b) represents the evolution of '$E_1$ & $E_2$' of the model. Plot (c) represents the evolution of '$I_1$ & $I_2$' of the model. Plot (d) represents the evolution of '$R_1$ & $R_2$' of the model. Plot (e) represents the evolution of '$I_3$ & $R_3$' of the model. Here DE in the label denotes the condition $\beta_1\tau_1\Lambda < \mu(\tau_1 + \rho\delta + \mu)(\sigma + \mu)$.

trol variables. Computationally identified results are shown in Fig. 5(a)-5(c) for each class $R_i$ $(i = 1, 2, 3)$ respectively. It is observed that the recovery population increases with the control variable; see Fig. 5.

# 7. Conclusion

A co-infection of a human or a pig with human influenza or COVID-19 strains and H5N1 strain may result in a pandemic strain, causing a widespread deadly pandemic. So, in this paper we consider the new model for co-infection of two pathogen strains such as COVID-19 (rapid virus) and HIV (slow virous) diseases. First, the model and its parameters are introduced in a detailed manner. Then, the wellposedness (Loosely speaking, a differential equation model such as the model (1.1) is well posed if through every point (initial condition), there exists a unique solution.) of the model was studied with the non-negativity and boundedness of the solution variables. Further, the basic reproduction number and the local stability for all possible equilibrium points were derived. Also, a sensitivity analysis of the model parameters was performed. A control problem was introduced for the proposed

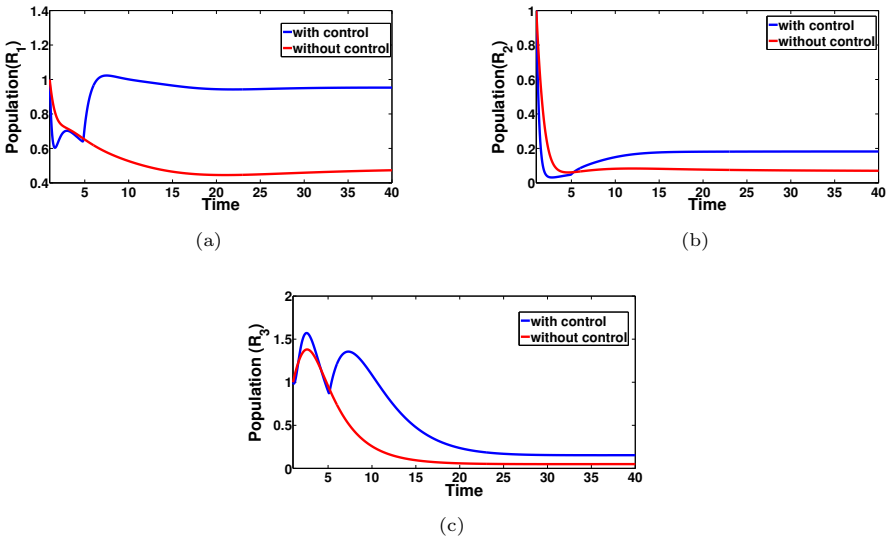


**Figure 5.** Plot (a) shows the change in the recovery population ($R_1$) of pathogen-1 (COVID-19) of the model (1.1). Plot (b) shows the change in the recovery population ($R_2$) of pathogen-2 (HIV) of the model (1.1). Plot (c) shows the change in the recovery population ($R_3$) of co-infection model.

**Table 3.** Parameter values for the optimal control (1.1)

| $u_{min}$ | $u_{max}$ | $\rho_1$ | $\rho_2$ | $\rho_3$ | $\epsilon_1$ | $\epsilon_2$ | $\epsilon_3$ |
|---|---|---|---|---|---|---|---|
| 0 | 0.9 | 0.2 | 0.5 | 0.3 | 0.1 | 0.2 | 0.15 |

model and analyzed with numerical computations.

# Acknowledgements

The authors would like to thank the referees for their suggestions and helpful comments which improved the presentation of the original manuscript.

**Conflict of interest:** The authors declare that they have no conflict of interest.

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