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Positive Periodic Solutions for First-Order Nonlinear Neutral Differential Equations

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Abstract We consider a first-order nonlinear neutral differential equation. By employing Krasnoselskii's fixed point theorem, we provide several new criteria for the existence of positive periodic solutions to this equation. The theorems we have formulated are exemplified through a specific example.

Keywords Fixed point, neutral equations, positive periodic solution, first-order

MSC(2010) 34C25, 34K13.

1. Introduction

In this current study, we explore the existence of positive ω-periodic solutions for a first-order neutral differential equation given by

$$(a(t)x(t))' = -b(t)x(t) + c(t)x'(t-h(t)) + f(t, x(t-h(t))), \tag{1.1}$$

where $a \in C^1(\mathbb{R}, (0,\infty))$, $b \in C(\mathbb{R}, (0,\infty))$, $c \in C^1(\mathbb{R}, \mathbb{R})$, $h \in C^2(\mathbb{R}, (0,\infty))$ with $h'(t) \neq 1$ for all $t \in [0, \omega]$, which are ω-periodic functions. Additionally, $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is an ω-periodic function in t, and ω is a positive constant.

In fact, neutral differential equations and periodic phenomena appear in different models from real world applications; please see, e.g., [1, 7, 9, 13]. Our investigation builds upon the positive periodic solutions of the equation

$$x'(t) = -a(t)x(t) + c(t)x'(t-g(t)) + q(t, x(t-g(t))), \tag{1.2}$$

with $0 \leqslant \frac{c(t)}{1-g'(t)} < 1$, $-1 \leqslant \frac{c(t)}{1-g'(t)} \leqslant 0$, initiated in [18]. Our study extends and generalizes the results from [18] by considering the special case when $a(t) = 1$, leading to the equation above. This indicates that our findings not only encompass, but also offer broader insight compared to those obtained in [18], particularly for the more general equation.

In summary, our study provides generalizations and new criteria for positive periodic solutions in (1.1), complementing existing research in [2, 3, 5, 6, 8, 10–12, 14–17, 19] that explores positive periodic solutions in various types of first-order neutral differential equations. Additionally, the work in [4] focuses on positive periodic solutions to second-order neutral differential equations.

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2. Main results

Consider a function space Φ_ω consisting of ω-periodic continuous functions equipped with the supremum norm, denoted as $\|x\| = \sup_{t\in[0,\omega]} |x(t)|$. It is evident that the pair $(\Phi_\omega, \|\cdot\|)$ forms a Banach space. Let $a_0 = \min_{t\in[0,\omega]} a(t)$ and $a_1 = \max_{t\in[0,\omega]} a(t)$.

Theorem 2.1. *Let $0 \leqslant c_0 \leqslant \frac{c(t)}{a(t)(1-h'(t))} \leqslant c_1 < 1$. Furthermore, assume that there exist positive constants m_0 and m_1 with $m_0 < m_1$ such that*

$$(1-c_0)a_1 m_0 \leqslant \frac{a(t)}{b(t)}\Big(f(t,x) - r(t)x\Big) \leqslant (1-c_1)a_0 m_1 \tag{2.1}$$

for all $t \in [0,\omega]$, $x \in [m_0, m_1]$, where

$$r(t) = \frac{\Big(c'(t) + \frac{b(t)}{a(t)}c(t)\Big)\Big(1-h'(t)\Big) + h''(t)c(t)}{\Big(1-h'(t)\Big)^2}.$$

Then, (1.1) has at least one positive ω-periodic solution $x(t) \in [m_0, m_1]$.

Proof. Clearly, obtaining an ω-periodic solution of (1.1) is equivalent to finding an ω-periodic solution for the following integral equation

$$\begin{aligned} x(t) = \frac{1}{a(t)}\Big[&\frac{c(t)}{1-h'(t)}x(t-h(t)) \\ &+ \int_t^{t+\omega} G(t,s)\Big[f(s,x(s-h(s))) - r(s)x(s-h(s))\Big]ds\Big], \end{aligned}$$

where

$$G(t,s) = \frac{e^{\int_t^s \frac{b(u)}{a(u)}du}}{e^{\int_0^\omega \frac{b(u)}{a(u)}du} - 1}.$$

Consider the set $\Phi = \{x \in \Phi_\omega : m_0 \leqslant x(t) \leqslant m_1, t \in [0,\omega]\}$, which forms a bounded closed and convex subset of Φ_ω. Now, define the operators $\mathcal{T}, \mathcal{S} : \Phi \to \Phi_\omega$ as follows:

$$(\mathcal{T}x)(t) = \frac{c(t)}{a(t)(1-h'(t))}x(t-h(t)) \tag{2.2}$$

and

$$(\mathcal{S}x)(t) = \frac{1}{a(t)}\int_t^{t+\omega} G(t,s)\Big[f(s,x(s-h(s))) - r(s)x(s-h(s))\Big]ds. \tag{2.3}$$

For every $x \in \Phi$ and $t \in \mathbb{R}$, deducing from (2.2) and (2.3), it becomes evident that

$$\begin{aligned} (\mathcal{T}x)(t+\omega) &= \frac{c(t+\omega)}{a(t+\omega)(1-h'(t+\omega))}x(t+\omega-h(t+\omega)) \\ &= \frac{c(t)}{a(t)(1-h'(t))}x(t-h(t)) \\ &= (\mathcal{T}x)(t) \end{aligned}$$