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Convergence Analysis for the SM-Iteration in Banach Spaces

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Abstract In this article, the fixed point of the SM iterative approach is approximated via Suzuki mapping. In the context of Banach spaces, we provide both weak and strong convergence for the SM iteration. Then, by comparing it with certain well-known iterations, we give some numerical examples to show the effectiveness of SM iteration for the Suzuki-type mapping.

Keywords Suzuki mapping, Opial's condition, SM-iteration

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1. Introduction

Fixed point theory is a valuable tool for resolving a variety of practical mathematics difficulties. The Banach contraction principle [5] was the first step toward the fixed point theory on metric spaces. Since many nonlinear analysis problems cannot be solved analytically, iterative approaches for approximating fixed points of various kinds of mappings become essential. In this sense, the development of other processes benefited greatly from the foundation provided by Picard iteration [14]. Though it was successful for contraction mappings, a wider class of mappings constructed on Banach spaces (BS), namely non-expansive mappings, may not necessarily converge to the fixed point, as demonstrated in 1955 by Krasnoselskii [10], where a mapping $\mathfrak{F} : V \to V$, for V being a non-empty closed and convex subset of BS E, is said to be non-expansive if it satisfies the inequality $\|\mathfrak{F}\varkappa - \mathfrak{F}y\| \leq \|\varkappa - y\|$, for all $\varkappa, y \in V$. In addition, if $Fix(\mathfrak{F}) \neq \phi$, where $Fix(\mathfrak{F}) = \{\varkappa \in V : \mathfrak{F}\varkappa = \varkappa\}$ and $\|\mathfrak{F}\varkappa - q\| \leq \|\varkappa - q\|$, for every $\varkappa \in V$, set of fixed points (FP), and $q \in Fix(\mathfrak{F})$, then $\mathfrak{F}$ is called quasi-non-expansive. The main reason for this behaviour is that successive iterations of non-expansive mappings do not need to converge to a fixed point, unlike contraction mappings. Since then, numerous additional iterative procedures have been developed for numerically calculating the fixed points of non-expansive mappings. For instance, one of the first iteration was proposed by Mann [11], which is described as follows: for an arbitrary chosen $\varkappa_0 \in V$, the iteration is defined as:

$$\varkappa_{n+1} = \alpha_n \varkappa_n + (1 - \alpha_n)\mathfrak{F}\varkappa_n, \quad n \geq 0,$$

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where, $\{\alpha_n\}$ is a real sequence in the interval $(0,1)$. After that two-step iterative method, which is Ishikawa [9] iteration, mostly used for finding fixed point of non-expansive mappings: for initial point $\varkappa_0 \in V$, this iteration is defined as

$$\begin{aligned} \varkappa_{n+1} =& (1-\alpha_n)\varkappa_n + \alpha_n \mathfrak{S} y_n, \\ y_n =& (1-\beta_n)\varkappa_n + \beta_n \mathfrak{S}\varkappa_n, \quad n \geq 0, \end{aligned}$$

where $\{\alpha_n\}, \{\beta_n\} \in (0,1)$. In the similar manner, Agarwal et al. [2], Noor [12], Abbas and Nazir [1] worked in the same direction. The subsequent three-step iterations, as described by Thakur et al. [19] in 2016, Rathee and Swami [15] in 2020, Ahmad et al. [4] in 2021, will be examined in this follow-up for finding fixed points of non-expansive mapping, which are defined, for self map $\mathfrak{S}$ on V, the initial point $\varkappa_0 \in V$ with real number sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0,1)$.

In 2016, Thakur [19] defined the following iteration:

$$\begin{aligned} \varkappa_{n+1} &= (1-\alpha_n)\mathfrak{S} z_n + \alpha_n \mathfrak{S} y_n, \\ y_n &= (1-\beta_n) z_n + \beta_n \mathfrak{S} z_n, \\ z_n &= (1-\gamma_n)\varkappa_n + \gamma_n \mathfrak{S}\varkappa_n. \end{aligned} \tag{TH}$$

In 2020, Rathee and Swami [15] proposed an iteration as follows:

$$\begin{aligned} \varkappa_{n+1} &= \mathfrak{S}((1-\alpha_n)\mathfrak{S} z_n + \alpha_n \mathfrak{S} y_n), \\ y_n &= \mathfrak{S}((1-\beta_n)\varkappa_n + \beta_n z_n), \\ z_n &= \mathfrak{S}\varkappa_n. \end{aligned} \tag{SM}$$

In 2021, the following iteration is defined by Ahmad et al. [4]

$$\begin{aligned} \varkappa_{n+1} &= \mathfrak{S}((1-\alpha_n)\mathfrak{S} z_n + \alpha_n \mathfrak{S} y_n), \\ y_n &= \mathfrak{S} z_n, \\ z_n &= (1-\beta_n)\varkappa_n + \beta_n \mathfrak{S}\varkappa_n. \end{aligned} \tag{JK}$$

On the other hand, in 2008, Suzuki [18] made significant progress by defining an intriguing expansion of non-expansive mappings, which is called Suzuki mapping (condition (C)). Suzuki mapping is defined for a self map $\mathfrak{S} : V \to V$ if the following condition holds:

$$\begin{aligned} \frac{1}{2}\|\varkappa - \mathfrak{S}\varkappa\| &\leq \|\varkappa - y\| \\ \implies \|\mathfrak{S}\varkappa - \mathfrak{S} y\| &\leq \|\varkappa - y\|, \quad \text{for all } \varkappa, y \in V. \end{aligned} \tag{1.1}$$

It is evident that for certain domain elements, the Suzuki mappings meet the non-expansive criteria. Suzuki [18] used a simple example to illustrate his point that the newly introduced class is larger than the class of non-expansiveness. For this, define a mapping T on $[0,3]$ by

$$T\varkappa = \begin{cases} 0 \text{ if } \varkappa \neq 3 \\ 2 \text{ if } \varkappa = 3. \end{cases}$$

This condition (C) caught the attention of many researchers who worked in the direction of finding different fixed point theorems. Here, in the current research,