Journal of Nonlinear Modeling and Analysis
Volume 7, Number 3, June 2025, 1125–1141
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.1125

Traveling Wave Solutions of Some *abcd*-Water Wave Models Describing Small Amplitude, Long Wavelength Gravity Waves on the Surface of Water*

Jibin Li[1,2†] and Zhilong Shi[3]

Abstract For some *abcd*-water wave models describing small amplitude, long wavelength gravity waves on the surface of water, in this paper, by using the method of dynamical systems to analyze corresponding traveling wave systems and find the bifurcations of phase portraits, the dynamical behavior of systems can be derived. Under some given parameter conditions, for a wave component, the existence of periodic wave solutions, solitary wave solutions, kink and anti-kink wave solutions as well as compacton families can be proved. Possible exact explicit parametric representations of the traveling wave solutions are given.

Keywords Pseudo-peakon, solitary wave, kink and anti-kink wave, compacton family, planar dynamical system, *abcd*-water wave models

MSC(2010) 34C23, 34C37, 35Q55, 37K45, 58Z05, 74J30.

1. Introduction

More recently, in [5], the authors studied exact Jacobi elliptic solutions for the following *abcd*-system:

$$\begin{aligned} &\eta_t + w_x + (w\eta)_x + aw_{xxx} - b\eta_{xxt} = 0, \\ &w_t + \eta_x + ww_x + c\eta_{xxx} - dw_{xxt} = 0, \end{aligned} \tag{1.1}$$

where a, b, c, and d are real constants and $\theta \in [0, 1]$ that satisfy

$$a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \quad c + d = \frac{1}{2}(1 - \theta^2), \quad a + b + c + d = \frac{1}{3}. \tag{1.2}$$

System (1.1) was introduced by Bona et al. in [1] and [2] to describe the wave motion of small amplitude, long wavelength gravity waves on the surface of water. The functions $\eta(x, t)$ and $w(x, t)$ are real valued and $x, t \in R$.

†The corresponding author.
Email address: lijb@zjnu.cn(Jibin Li)
[1]School of Mathematical Sciences, Zhejiang Normal University, Jinhua, Zhejing 321004, P. R. China
[2]School of Mathematical Science, Huaqiao University, Quanzhou, Fujian 362021, P. R. China
[3]Faculty of Architectural Engineering, Kunming University of Science and Technology, Kunming, Yunnan 650500, China
*This research was partially supported by the National Natural Science Foundations of China (11871231, 12071162, 11701191).

A traveling-wave solution to the system (1.1) is a vector solution $(\eta(x,t), w(x,t))$ of the form

$$\eta(x,t) = \phi(x-\sigma t) = \phi(\xi),\ w(x,t) = \psi(x-\sigma t) = \psi(\xi),\ \xi = x - \sigma t, \tag{1.3}$$

where σ denotes the speed of the waves.

Substituting (1.3) into (1.1) and integrating the obtained system once, we have

$$\begin{aligned} &-\sigma\phi + \psi + \phi\psi + a\psi'' + b\sigma\phi'' = g_1, \\ &-\sigma\psi + \phi + \frac{1}{2}\psi^2 + c\phi'' + d\sigma\psi'' = g_2, \end{aligned} \tag{1.4}$$

where "′" denotes the derivative in ξ, g_1 and g_2 are two integral constants. System (1.4) can be written as:

$$\begin{aligned} \left(1 - \frac{\psi}{\sigma}\right)\phi - b\phi'' &= g_1 + \frac{1}{\sigma}(a\psi'' + \psi), \\ \phi + c\phi'' &= g_2 - d\sigma\psi'' - \frac{1}{2}\psi^2 + \sigma\psi. \end{aligned} \tag{1.5}$$

(1.5) implies that if $c \neq 0$ and $a^2 + b^2 \neq 0$,

$$\phi = F(\psi, \psi'') = \frac{b(g_2 - d\sigma\psi'' - \frac{1}{2}\psi^2 + \sigma\psi) + c(g_1 + \frac{1}{\sigma}(a\psi'' + \psi))}{(b+c) - \frac{c}{\sigma}\psi}. \tag{1.6}$$

Notice that

$$\frac{d\phi}{d\xi} = \frac{\partial F}{\partial \psi}\psi' + \frac{\partial F}{\partial \psi''}\psi''', \quad \frac{d^2\phi}{d\xi^2} = \frac{\partial^2 F}{\partial \psi^2}(\psi')^2 + \frac{\partial F}{\partial \psi}\psi'' + \frac{\partial^2 F}{(\partial \psi'')^2}(\psi''')^2 + \frac{\partial F}{\partial \psi''}\psi''''. \tag{1.7}$$

Generally, substituting (1.6) and (1.7) into the second equation of (1.5), we obtain a fourth order ordinary differential equation about the variable ψ, because it contains a fourth order derivative ψ'''' with respect to ξ.

If $c \neq 0, a = b = 0$ or $a = d = 0$, then, we can obtain a second order traveling wave differential equation.

Obviously, to investigate the traveling wave solutions of the PDE system (1.1), we must study the all solutions of the corresponding ordinary differential equation (traveling system). [5] did not discuss the dynamics of solutions for the traveling system of system (1.1). Therefore, the conclusions in their paper are not complete and some results are incorrect.

By choosing specific values for the parameters a, b, c, and d, the system (1.1) includes a wide range of other systems that have been derived over the last few decades such as the classical Boussinesq system, the Kaup system, the coupled Benjamin-Bona-Mahony system (BBM-system), the coupled Korteweg-de Vries system (KdV-system), the Bona-Smith system, and the integrable version of Boussinesq system. In particular, these specializations are (see Bona et al. [1]):

(i) Classical Boussinesq system $(\theta^2 = \frac{1}{3}, a = b = c = 0, d = \frac{1}{3})$:

$$\begin{aligned} &\eta_t + w_x + (w\eta)_x = 0, \\ &w_t + \eta_x + ww_x - \frac{1}{3}w_{xxt} = 0. \end{aligned} \tag{1.8}$$