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Entire Solutions for Certain Class of Non-Linear General Difference Equations

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Abstract In this paper, we investigate the entire solutions for a certain class of non-linear difference equations of the form: $f^n + q(z)e^{Q(z)}\mathcal{L}_1(z,f) = \alpha_1(z)e^{\beta_1(z)} + \alpha_2(z)e^{\beta_2(z)}$, where $\mathcal{L}_1(z,f)$ is the generalized linear difference operator, $\alpha_1(z)$ and $\alpha_2(z)$ are non-zero small functions of f, $q(z)$ and $Q(z)$(non-constant), $\beta_1(z)$ and $\beta_2(z)$ are non-zero polynomials. Our results improve upon and generalize some previously established findings.

Keywords Non-linear difference equations, meromorphic function, entire solution, Nevanlinna theory

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1. Background information

Assuming the reader's familiarity with conventional notations and core outcomes of Nevanlinna's theory on meromorphic functions (see [9]), in this paper, we consistently refer to meromorphic functions as those meromorphic in the entire complex plane $\mathbb{C}$. For a meromorphic function f and $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, any z such that $f(z) = a$ is termed an a-point of f. In 1926, the Finnish mathematician Rolf Nevanlinna made a noteworthy breakthrough in complex analysis by investigating meromorphic functions over the complex plane. He demonstrated that a non-constant function can be uniquely determined by five distinct pre-images, including infinity, without considering multiplicities. This finding is particularly interesting because it has no counterpart in the real function theory. Later, Nevanlinna went on to prove that when multiplicities are taken into account, four points are adequate for determining the uniqueness of a pair of meromorphic functions. In such cases, either the functions coincide, or one is a bilinear transformation of the other. These seminal discoveries marked the beginning of research into the uniqueness of pairs of meromorphic functions, especially when one function is related to the other. Two meromorphic functions $f(z)$ and $g(z)$ share a CM(Counting multiplicity) or IM(Ignoring multiplicity) if $f - a$ and $g - a$ have the same set of zeros counting multiplicities or ignoring multiplicities, respectively. Further recall that the order of f is defined by

$$\rho(f) = \lim_{r\to\infty} \sup \frac{\log T(r,f)}{\log r}.$$

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Establishing the existence of solutions for complex differential equations represents a significant and challenging problem. Nevanlinna theory has found extensive application in analyzing the properties of such equations. In recent times, an increasing number of researchers have employed Nevanlinna theory to study the solutions of complex differential equations. Moreover, certain topics related to complex difference equations or complex nonlinear differential-difference equations have also been explored using the difference analogs of Nevanlinna theory (see [16], [10], [17]). Notably, in 1964, Hayman [9] examined the behavior of nonlinear differential equations of the following form:

$$f^n + P_d(z, f) = g(z), \tag{1.1}$$

where $P_d(z, f)$ is a differential polynomials in f of degree d with meromorphic coefficients of growth $S(r, f)$ and $n \geq 2$ is an integer.

In 2004, C.C. Yang and P. Li [20] demonstrated that the differential equation $4f^3 + 3f'' = -\sin 3z$ possesses precisely three non-constant entire solutions, namely $f_1(z) = \sin(z)$, $f_2(z) = \frac{\sqrt{3}}{2}\cos(z) - \frac{1}{2}\sin(z)$, and $f_3(z) = -\frac{\sqrt{3}}{2}\cos(z) - \frac{1}{2}\sin(z)$. Since $\sin(3z)$ can be expressed as a linear combination of e^{3iz} and e^{-3iz}, this result has stimulated the interest of numerous scholars to investigate the more general differential equation

$$f^n + P_d(z, f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z},$$

where $P_d(z, f)$ is a polynomial in f and its derivatives with meromorphic coefficients. Subsequently, it was demonstrated in [19] that the equation:

$$f^2 + q(z)f(z + 1) = p(z),$$

where $p(z)$ and $q(z)$ are polynomials, does not admit any transcendental entire solutions of finite order. As a result of the interest generated by the initial findings, numerous investigations have been undertaken by examining various forms of the function $g(z)$ in the non-linear differential equation (1.1). For a more comprehensive overview and additional details regarding non-linear differential equations, one may consult [1, 11, 15, 23, 24].

Several authors have been intrested in investigating the solution of the following type of equation

$$f^n + P_d(z, f) = \alpha_1(z)e^{\beta_1(z)} + \alpha_2(z)e^{\beta_2(z)}, \tag{1.2}$$

where $P_d(z, f)$ is a differential polynomials in f of degree d and $\alpha_1(z)$, $\alpha_2(z)$, $\beta_1(z)$ and $\beta_2(z)$ are polynomials. Several works pertinent to the topics discussed can be found in [3, 4, 7, 12, 22]. For instance, Liu et al. [14], in their study referenced therein, investigated the existence of meromorphic solutions for the equation (1.2) and derived the following result.

Theorem 1.1. *[14] Let $n \geq 3$ be an integer and $d \leq n - 2$ be the degree of differential polynomial $P_d(z, f)$. Consider the polynomials $\beta_1(z)$, $\beta_2(z)$ of degree $k(\geq 1)$ and $\alpha_1(z)$, $\alpha_2(z)$ be two small non-zero meromorphic functions of e^{z^k}. If $\frac{\beta_1^{(k)}}{\beta_2^{(k)}} \notin \left\{\frac{n}{n-1}, \frac{n-1}{n}, -1, 1\right\}$, and any one of the following occurs*

(i) $P_d(z, f) \not\equiv 0$,

(ii) $P_d(z, f) \equiv 0$, $\frac{\beta_1^{(k)}}{\beta_2^{(k)}} \notin \left\{\frac{n}{d}, \frac{d}{n}\right\}$,