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Strong Convergence Theorem Involving Two-Step Inertial Technique Without On-Line Rule for Split Feasibility Problem

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Abstract This work presents an approach for solving the split feasibility problem in an efficient manner. For solving the split feasibility problem, we present a method with a two-step inertial extrapolation and self-adaptive stepsize. The adjustable stepsize and two-step inertial extrapolation both contribute to the proposed method's improved rate of convergence and decreased computational complexity. The strong convergence results are obtained without on-line rule of the inertial parameters and the iterates. This makes our proof arguments different from what is obtainable in the literature where online rule is imposed on algorithms involving inertial extrapolation step. As far as we know, no strong convergence result has been obtained before now for algorithms with two step inertial for solving split feasibility problems in the literature. To demonstrate the viability of our suggested strategy, numerical results are provided at the end.

Keywords Split feasibility problems, two-step inertial technique, CQ methods, strong convergence

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1. Introduction

The split feasibility problem (SFP) is defined as the problem of finding a point $\hat{x} \in \mathcal{C}$ such that

$$A\hat{x} \in \mathcal{Q}, \tag{1.1}$$

where $\mathcal{C} \subseteq \mathcal{H}_1$ and $\mathcal{Q} \subseteq \mathcal{H}_2$ are nonempty, closed and convex sets, and $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded and linear operator. We denote the solution set of the problem (1.1) by Ω. For the purpose of resolving inverse problems associated with phase retrievals and medical image recovery, Censor and Elfving [10] first presented the SFP in

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finite dimensional Hilbert space. Numerous studies have demonstrated the SFP's versatility in a variety of fields, including computer tomography, picture restoration, and data reduction (see [11–14, 33, 39] and other references therein).

Various iterative strategies for solving the SFP have been investigated and introduced by a number of researchers (see [9, 15–17, 20, 36, 41] and other references therein). The CQ algorithm, developed by Bryne [8], is a well-known algorithm for finding the solution to the SFP. Iteratively, this algorithm generates the sequence $\{x_k\}$:

$$x_{k+1} = P_{\mathcal{C}}(x_k - \lambda A^*(I - P_{\mathcal{Q}})Ax_k), \quad \forall\ k \geq 1, \tag{1.2}$$

where $\lambda \in \left(0, \frac{2}{\|A\|^2}\right)$, $P_{\mathcal{C}},\ P_{\mathcal{Q}}$ are the orthogonal projections onto $\mathcal{C}$ and $\mathcal{Q}$ respectively. A weak convergence result was established by the author. The drawback of this approach is that it requires the calculation of the spectral radius of the matrix A^*A or the norm estimate of the linear operator A, both of which are challenging and occasionally impossible to do in an infinite dimensional setting. To overcome this drawback, Byrne [8] presented a method for estimating matrix norms (see [8], Proposition 4.1). The condition on this method is highly stringent. In order to overcome this drawback, López et al. [27] substituted an adaptive stepsize for the stepsize in (1.2) and defined it as follows:

$$\eta_k = \frac{\sigma_k F(x_k)}{\|\nabla F(x_k)\|^2}, \quad k \geq 1,$$

where $\eta_k \in (0, 4)$, $F(x_k) = \frac{1}{2}\|(I - P_{\mathcal{Q}})Ax_k\|^2$ and $\nabla F(x_k) = A^*(I - P_{\mathcal{Q}}Ax_k)$ for all $k \geq 1$. Several authors have adopted this adaptive stepsize for solving the SFP (see [19, 20, 25]).

Iterative methods for approximating solutions of the SFP are known to have slow convergence properties. In recent years, a host of researchers have invested considerable effort into enhancing the convergence properties of these iterative algorithms. Among the prominent strategies for getting acceleration is the inertial extrapolation technique, which traces its roots back to Polyak's early work on smooth convex minimization problems. In essence, the inertial acceleration strategy entails forming a nonconvex combination of two previous terms to derive the subsequent iterate. For further insights into this technique and its applications to iterative methods tailored for solving the SFP (1.1), interested readers may refer to [2, 3, 5, 6, 32], along with the additional references cited therein.

Dang et al. [18] recently proposed one-step inertial relaxed CQ techniques for finding the solution of (1.1) and they proposed them as follows:

$$x_{k+1} = P_{\mathcal{C}_k}(y_k - \lambda A^*(I - P_{\mathcal{Q}_k})Ay_k) \tag{1.3}$$

and

$$x_{k+1} = (1 - \alpha_k)y_k + \alpha_k P_{\mathcal{C}_k}(y_k - \lambda A^*(I - P_{\mathcal{Q}_k})Ay_k), \tag{1.4}$$

where $y_k = x_k + \theta_k(x_k - x_{k-1})$, $\alpha_k \in (0, 1)$, $\lambda \in \left(0, \frac{2}{\|A\|^2}\right)$ and $0 \leq \theta_k \leq \bar{\theta}_k$ with

$$\bar{\theta}_k := \min\left\{\theta, \left(\max\frac{1}{\{k^2\|x_k - x_{k-1}\|^2,\ k^2\|x_k - x_{k-1}\|\}}\right)\right\}, \quad \theta \in [0, 1).$$

The authors proved that $\{x_k\}$ generated by algorithms (1.3) and (1.4) converges weakly to a point in Ω.