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Mild Solution for the Time Fractional Hall-Magneto-Hydrodynamics Stochastic Equations

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Abstract In this paper, we establish the existence and uniqueness of mild solutions for the time fractional hall-magneto-hydrodynamics stochastic equations with a fractional derivative of Caputo. Initially, we focus on the existence and uniqueness in the deterministe case. Using the Mittag-Leffler operators $\{\mathcal{Q}_\alpha(-t^\alpha \mathbb{J}) : t \geq 0\}$ and $\{\mathcal{Q}_{\alpha,\alpha}(-t^\alpha \mathbb{J}) : t \geq 0\}$ and applying the bilinear fixed-point theorem, we will prove the frist result. Next, by Itô integral, and by similair analogy we will establish the existence and uniqueness in the stochastic case in $\mathcal{EN}_a^\mu \cap \mathrm{N}_{a,\mu}^{2\alpha}$.

Keywords Time fractional hall-magneto-hydrodynamics equations, Itô integral, derivative of Caputo, stochastic

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1. Introduction

The hall-magneto-hydrodynamics equations (HMHD) describes the evolution of a system consisting of charged particles that can be approximated as a conducting fluid. The HMHD equation is given by:

$$\begin{cases} v_t + (v\cdot\nabla)v + \mu(-\Delta)^\beta v + \nabla\pi = (b\cdot\nabla)b & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ b_t + (v\cdot\nabla)b + \nu(-\Delta)^\gamma b + \nabla\times((\nabla\times b)\times b) = (b\cdot\nabla)v & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ \nabla\cdot v = 0, \quad \nabla\cdot b = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ v|_{t=0} = v_0,\, b|_{t=0} = b_0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $v = (v_1, v_2, v_2)$ represents the velocity field of the flow, $b = (b_1, b_2, b_3)$ denotes the magnetic field, π denotes the pressure function, $\mu > 0$ denotes the viscosity

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coefficient, and ν represents the diffusivity coefficient, v_0 and b_0 are respectively the initial velocity and the initial magnetic field with free divergence (i.e $\nabla\cdot v_0 = 0$ and $\nabla\cdot b_0 = 0$). The operator $(-\Delta)^\beta$ is the Fourier multiplier of symbol $|\mathcal{K}|^{2\beta}$ given by

$$\mathcal{F}\big((-\Delta)^\beta v\big) = |\mathcal{K}|^{2\beta}\mathcal{F}(v),$$

where $\mathcal{F}$ is the Fourier transform. To simplify and without loss of generality, we consider only the case where $\mu = \nu = 1$.

The fractional calculus has a long history, going back to the early days of differential calculus. Many mathematicians like Abel, Liouville, Euler, Riemann, Leibniz, l'Hôpital, and Fourier have discussed and studied it. In the last forty years, several rechearchers have studied it deeply and made amazing discoveries. At first, it was seen as something abstract and not useful in the real world. But recently, scientists have found that it can be applied to many different domains of science. This new usefulness comes from the unique way fractional calculus deals with nonlocal characteristics.

Much work has been done on fractional calculus, please refer to the complete study [9] and associated references. For instance, regarding its applications in physics, more specifically in electromaginetism, see [7, 21] and for viscoelasticity see [1, 2, 5, 18, 26]. Yimin Xiao in [29] gives an important application to stochastic processes induced by fractional Brownian motion. For further examples, see the extensive survey [9, 27] and the references therein.

In this respect, and by the same reasoning as Shinbrot [20], we can show some lemmas about the regularity of the fractional derivative of HMHD equations.

So it is not surprising to start studying this topic by proposing hall-magneto-hydrodynamics equations with a time fractional differential operator in time:

$$\begin{cases} {}^c\mathrm{D}_t^\alpha v + (v\cdot\nabla)v + \mu(-\Delta)^\beta v + \nabla\pi = (b\cdot\nabla)b & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ {}^c\mathrm{D}_t^\alpha b + (v\cdot\nabla)b + \nu(-\Delta)^\gamma b + \nabla\times((\nabla\times b)\times b) = (b\cdot\nabla)v & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ \nabla\cdot v = 0, \quad \nabla\cdot b = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ v|_{t=0} = v_0,\, b|_{t=0} = b_0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where ${}^c\mathrm{D}_t^\alpha$ is the Caputo fractional derivative of order $\alpha \in (0,1)$ defined by

$${}^cD_t^\alpha\omega(t) := \frac{d}{dt}\left\{W_t^{1-\alpha}[\omega(t)-\omega(0)]\right\} = \frac{d}{dt}\left\{\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\rho)^{-\alpha}[\omega(\rho)-\omega(0)]ds\right\},$$

where $W_t^\alpha\omega(t)$ is the Riemann-Liouville fractional integral of order α of a function $\omega \in L^1(0,T;X)$ given by

$$W_t^\alpha\omega(t) := f_\alpha * \omega(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega(s)ds, \quad t\in[0,T],$$

and $\Gamma(\alpha)$ is the Euler's Gamma function for any positive value of α while g_α is defined as follows:

$$g_\alpha(t) := \begin{cases} \frac{1}{\Gamma(\alpha)}t^{\alpha-1}, & t>0, \\ 0, & t\le 0. \end{cases}$$