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An Inertial Tseng's Extragradient Method for Approximating Solution of Split Problems in Banach Spaces

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Abstract In this paper, we introduce a new inertial type algorithm with a self-adaptive step size for approximating a common element of the set of solutions of split common null point and pseudomonotone variational inequality problem as well as the set of common fixed point of a finite family of quasi nonexpansive mappings in uniformly smooth and 2-uniformly convex real Banach space. The proposed algorithm is constructed in such a way that its convergence analysis does not require a prior estimate of the operator norm. We also give numerical examples to illustrate the performance of our algorithm. Our results generalize and improve many existing results in the literature.

Keywords Variational inequality problem, inertial Tseng's extragradient method, fixed point, Banach spaces

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1. Introduction

Let E be a real Banach space and E^* be its dual space. Let C be a nonempty, closed and convex subset of E, and let $F : C \to E^*$ be a mapping. The problem of finding a point $x^* \in C$ such that

$$\langle y - x^*, Fx^* \rangle \geq 0, \quad \forall\, y \in C, \tag{1.1}$$

is called a *variational inequality problem*. The set of solutions of variational inequality problem (1.1) is denoted by $VI(C, F)$. The study of variational inequality problem originates from solving minimization problems involving infinite-dimensional functions and calculus of variation (see, for example, [33] and reference therein). The concept of variational inequality problem was initially introduced by Hartman and Stampacchia [18] as a generalization of boundary value problems in 1966. Such problems are applicable in a wide range of applied sciences and mathematics. Later in 1967 Lions and Stampacchia [28] studied the existence and uniqueness of the solution. Since then, the theory of variational inequality problem has received

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much attention due to its wide applications in various areas of pure and applied sciences, such as optimal control, image recovery, resource allocations, networking, transportation, signal processing, game theory, operation research and so on (see, for example, [3,23,39] and references therein). The constraints can clearly be expressed as variational inequality problems and (or) as fixed point problems. Consequently, the problem of finding common elements of the set of solutions of variational inequality problems and the set of fixed points of nonlinear operators has become an interesting area of research for many researchers working in the area of nonlinear operator theory (see, for example, [30,31] and the references contained in them). In view of this, many researchers in their quest to find solutions of variational inequality problems have proposed and analyzed various iterative approximation methods (see for example, [13, 20])in which most of them are based on projection methods. The simplest and earliest form of projection method is due to Goldstein [17], which is a natural extension of the gradient projected technique considered for solving optimization problems. A number of results on iterative methods proposed for approximating solutions of variational inequality problems are studied such that the operator F was often considered to be either strongly monotone or inverse strongly monotone (see, for instance [17, 26] and references therein) for convergence to be guaranteed. In order to relax the strong monotonicity condition imposed on the operator F, Korpelevich [25] proposed the following extragradient method in a finite dimensional Euclidean space $\mathbb{R}^n$:

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C(x_n - \lambda F(x_n)), \\ x_{n+1} = P_C(x_n - \lambda F(y_n)) \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

where $\lambda \in (0, \frac{1}{L})$, F is monotone and Lipschitz and P_C is the metric projection onto C. They proved that the sequence $\{x_n\}$ generated by algorithm (1.2) converges weakly to a solution of problem (1.1). However, the extragradient method requires the computation at each step of the iteration process two projections onto an arbitrary closed and convex subset C of H. This might affect the efficiency of the extragradient method if the feasible set is not simple enough which might also increase the computational cost.

In order to overcome this barrier, several modifications of the extragradient method were proposed (see, for example [12,19,44] and references therein) for solving variational inequality problem (1.1). In particular, Tseng [44] proposed the following Tseng's extragradient method

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C(x_n - \lambda F(x_n)), \\ x_{n+1} = y_n - \lambda(F(y_n) - F(x_n)) \quad \forall n \geq 0, \end{cases} \tag{1.3}$$

where $\lambda \in (0, \frac{1}{L})$, F is monotone and Lipschitz and P_C is the metric projection onto C. They proved that the sequence $\{x_n\}$ generated by algorithm (1.3) converges weakly to a solution of problem (1.1) in a real Hilbert space. Another modification