Journal of Nonlinear Modeling and Analysis https://journal.global-sci.org/jnma
Volume 7, Number 4, August 2025, 1431–1445 DOI:10.12150/jnma.2025.1431

Local Bifurcation Cyclicity for a Non-Polynomial System*

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Abstract In this paper, we propose a class of general non-polynomial analytic oscillator models, and study the limit cycle bifurcation at the nilpotent singularity or elementary center-focus. By Taylor expansion, two specific systems from the original model are transformed into two equivalent infinite polynomial systems, and the highest order of fine focus as the nilpotent Hopf bifurcation or Hopf bifurcation point is determined respectively. At the same time, the local bifurcation cyclicities and center problems for two systems are solved respectively. To our knowledge, such dynamic properties are rarely analyzed in many non-polynomial models.

Keywords Non-polynomial system, quasi-Lyapunov constant, nilpotent singularity, Hopf bifurcation

MSC(2010) 34C05, 37C07.

1. Introduction

In this paper, we study the following weak perturbation nonlinear oscillator model:

$$\ddot{x} = -\kappa_1 \sin x + \kappa_2 \sin x \cos x + h(x, \dot{x}), \tag{1.1}$$

where $\kappa_1 > 0$, $\kappa_2 > 0$ and $h(x, \dot{x})$ as a perturbation part is any smooth function on $\mathbb{R}^2$. Its background comes from a classic case which often appears in college physics textbooks: a class of overdamped ball motion models on a rotating ring [24]. The trajectory of the ball can be described by the following second-order ordinary differential equation.

$$mr\ddot{x} = -b\dot{x} - mg \sin x + mr\omega^2 \sin x \cos x, \tag{1.2}$$

where x is the swing angle, $r\ddot{x}$ is the acceleration, mg is the gravity, $mr\omega^2 \sin x \cos x$ is the lateral centrifugal force, the $b\dot{x}$ is the tangential damping force. This model shows rich dynamics properties, including various bifurcations [24].

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*This work was partially supported by Natural Science Foundation of China (Nos.12161023, 12061016), Nature Science Foundations of Guangxi (No.2025GXNSFAA069254), School graduate program (2025YCXS125)

For the undamped case, i.e., when $b = 0$, the energy Hamilton function corresponding to model (1.2) can be written as follows:

$$H = \frac{\dot{x}^2}{2} - \frac{g}{r}\cos x + \frac{\omega^2}{2}\cos^2 x. \tag{1.3}$$

In fact, the existence of viscous damping in the actual background makes the target ball in model (1.2) be affected by nonlinear damping, which is similar to the Van der Pol oscillator model [5] and Rayleigh oscillator model [23], described by

$$\ddot{x} = \mu(1 - x^2)\dot{x} - x, \quad \text{and} \qquad \ddot{x} = \mu(1 - \dot{x}^2)\dot{x} - x,$$

respectively, where μ is a scalar parameter representing the strength of the damping. Therefore, the term of damping force $b\dot{x}$ can be extended in the model (1.2), and we can propose the general model (1.1).

Furthermore, letting $\dot{x} = y$, and taking $h(x, \dot{x})$ as a specific polynomial function, we can transform model (1.1) into the non-polynomial analytic system as follows,

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\kappa_1 \sin x + \kappa_2 \sin x \cos x + h(x, y), \end{cases} \tag{1.4}$$

where $h(x, y) = ax + by + \sum_{2 \le i+j \le n} b_{ij} x^i y^j$, and $a, b, b_{ij} \in \mathbb{R}$, $n \in \mathbb{Z}$. Motivated by many research works on the cyclicity of piecewise smooth systems or non-polynomial Hamiltonian systems, e.g. [11, 12, 27], we will study such a non-polynomial system on the local bifurcation cyclicity, that is, the maximum number of small amplitude limit cycles that can bifurcate in the vicinity of equilibrium.

For the trigonometrical functions in some practical models, the approximations: $\sin x \approx x$ and $\cos x \approx 1$ are usually used when x is small. However, such approximations are inappropriate in solving the local bifurcation cyclicity of the original system. It is easily checked that the Jacobian matrix of (1.4) at its origin

$$J = \begin{pmatrix} 0 & 1 \\ a + \kappa_2 - \kappa_1 & b \end{pmatrix} \tag{1.5}$$

has a pair of conjugate imaginary eigenvalues if and only if $b = 0, a + \kappa_2 - \kappa_1 < 0$, namely the origin is a Hopf bifurcation point. Furthermore, if and only if $b = 0$, and $a + \kappa_2 - \kappa_1 = 0$, the Jacobian matrix (1.5) has double zero eigenvalues, i.e., the origin is a nilpotent singularity.

In this paper, we consider the two above categories of equilibria for system (1.4). For the former, to solve its Hopf bifurcation cyclicity in a planar polynomial vector field of degree $n \ge 2$, the general approach is to determine the highest order of fine focus by computing the focal values or Lyapunov constants and finding the center conditions. There have been some classic methods, and the reader can see [13, 14, 17, 21, 25]. While the quadratic case has been completely solved, that is, Hopf bifurcation cyclicity 3 was proven by Bautin [4], while for $n > 2$, this problem is still open. More recent new progress can be found in [9, 26] and references therein. Here to overcome the difficulty arising from such non-polynomial functions, we will adopt Taylor expansion to transform the original model into its equivalent infinite polynomial systems, then apply the method proposed in [17] to calculate the focal values.