Journal of Nonlinear Modeling and Analysis https://journal.global-sci.org/jnma
Volume 7, Number 4, August 2025, 1482–1496 DOI:10.12150/jnma.2025.1482

Existence and Uniqueness Results for Solutions to Fractional $p(\cdot, \cdot)$-Laplacian Problems with a Variable-Order Derivative

Abdelilah Azghay[1,2,†], Mohammed Massar[1,3] and Abderrahim El Mhouti[2]

Abstract This paper investigates a class of fractional problems involving the variable-order $p(\cdot, \cdot)$-Laplacian with homogeneous Dirichlet boundary conditions. Under suitable assumptions on the nonlinear term, we establish novel existence and uniqueness results for weak solutions. We achieve this by combining variational techniques with a result from the theory of monotone operators. Additionally, we reveal several interesting properties of the solution.

Keywords Fractional $p(\cdot, \cdot)$-Laplacian, uniqueness, monotone operator theory, variational methods

MSC(2010) 35A15, 35D30, 35J35, 35J60.

1. Introduction

In recent decades, there has been a notable surge in interest and significance surrounding nonlinear problems that involve nonlocal and fractional pseudo-differential operators. The exploration of these problems has been motivated by their wide-ranging applications across various fields of applied sciences. These applications encompass physics and engineering, population dynamics, finance, chemical reaction design, optimization, minimal surfaces, and game theory (as detailed in references [9, 16, 17, 24]). Moreover, differential equations and variational problems with variable exponents have gained great attention due to their strong physical relevance. As evidenced in references [1, 13, 22], such equations emerge in the mathematical modeling of fluid dynamics, including electrorheological and thermorheological fluids. They are also encountered in elastic mechanics, image restoration, and biology (as indicated in references [11, 20, 21, 30]). Notably, recent research on fractional $p(x, \cdot)$-Laplacian problems and the corresponding variational problems can be found in references [2, 4–6, 10, 14, 15, 26, 27].

In this current paper, we focus on establishing the existence and uniqueness of

†the corresponding author.
Email address: azghay.abdelilah@etu.uae.ac.ma (A. Azghay)
[1]Department of Mathematics, FSTH, Abdelmalek Essaadi University, Tetouan, Morocco.
[2]ISISA, FS, Abdelmalek Essaadi University, Tetouan, Morocco.
[3]Department of Mathematics, FPK, Sultan Moulay Slimane University Beni Mellal, Morocco.

weak solutions for the subsequent problem:

$$\begin{cases} \left(-\Delta_{p(\cdot,\cdot)}\right)^{\kappa(\cdot,\cdot)} u + a(x)|u|^{p(x,x)-2}u = \mu f(x,u) \text{ in } \Omega, \\ \\ \qquad\qquad\qquad\qquad u = 0 \qquad \text{ in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{P_μ}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, the variable exponent $p(\cdot,\cdot) : \mathbb{R}^{2N} \to (1,\infty)$ and the variable fractional $\kappa(\cdot,\cdot)$-order $\kappa(\cdot,\cdot) : \mathbb{R}^{2N} \to (0,1)$, are continuous functions, with $N > \kappa(x,y)p(x,y)$ for all $(x,y) \in \mathbb{R}^{2N}$, they fulfill the following two conditions respectively:

$$p(\cdot,\cdot) \text{ is symmetric and } 1 < \inf_{(x,y)\in\mathbb{R}^{2N}} p(x,y) =: p^- \le \sup_{(x,y)\in\mathbb{R}^{2N}} p(x,y) =: p^+ < \infty, \tag{1.1}$$

$$\kappa(\cdot,\cdot) \text{ is symmetric and } 0 < \inf_{(x,y)\in\mathbb{R}^{2N}} \kappa(x,y) =: \kappa^- \le \sup_{(x,y)\in\mathbb{R}^{2N}} \kappa(x,y) =: \kappa^+ < 1, \tag{1.2}$$

$a : \overline{\Omega} \to [0,\infty)$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $\mu > 0$ is parameter and $\left(-\Delta_{p(x,\cdot)}\right)^{\kappa(x,\cdot)}$ denotes the variable fractional $\kappa(\cdot,\cdot)$-order fractional $p(\cdot,\cdot)$-Laplacian operator defined as

$$\left(-\Delta_{p(x,\cdot)}\right)^{\kappa(x,\cdot)} u(x) = p.v. \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+\kappa(x,y)p(x,y)}} dy,\ x \in \mathbb{R}^N,$$

where $p.v.$ is employed as an abbreviation in the principal value sense.

Note that when $\kappa(\cdot,\cdot) = \kappa$ (constant), $\left(-\Delta_{p(x,\cdot)}\right)^{\kappa(x,\cdot)}$ becomes the fractional $p(x,\cdot)$-Laplacian operator and problem (P_μ) reduces to fractional $p(x,\cdot)$-Laplacian problem studied by M. Ait Hammou [2]. By employing the Berkovits topological degree theory, the author proved the existence of at least one weak solution for (P_μ).

The variable-order fractional derivatives extend the concept of constant-order fractional derivatives, first proposed by S. G. Samko and B. Ross [23]. In this approach, the derivative order can vary continuously based on dependent or independent variables, allowing for a better representation of memory effects over time or space [7]. C. F. Lorenzo and T. T. Hartley later applied this concept to model diffusion processes that respond to temperature fluctuations [18], which can also be used to describe temperature changes [19].

Very recently, considerable attention has been focused by many researchers about the existence of at least one or multiple solutions for $p(x,\cdot)$-Laplacian problems in the fractional variable-order case see (for example [8, 28, 29, 31]). In [8], R. Biswas and S. Tiwari considered the following fractional nonlocal Choquard problem:

$$\begin{cases} (-\Delta)^{s(\cdot)}_{p(\cdot)} u(x) = \lambda |u(x)|^{\alpha(x)-2}u(x) + \left(\int_\Omega \frac{F(y,u(y))}{|x-y|^{\mu(x,y)}} dy\right) f(x,u(x)) \text{ in } \Omega, \\ \\ \qquad\qquad u = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{ in } \mathbb{R}^N \backslash \Omega, \end{cases}$$

under suitable assumption on α, μ, s, f and by using the variational approach, the authors have established the existence of at least two distinct nontrivial weak solutions to the problem. In [29], the existence and uniqueness of weak solutions to variable-order fractional $p(x,\cdot)$-Laplacian equation have been discussed.