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Subdifferential Frictional Contact Problem with Thermo-Electro-Visco-Elastic Locking Materials: Analysis and Approximation

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Abstract This paper investigates a frictional contact problem involving a thermo-electro-visco-elastic model for locking materials in contact with a rigid foundation. Friction is described by the subgradient of a locally Lipschitz function, while contact is governed by Signorini's unilateral condition. We formulate the problem as a system of three hemivariational inequalities and establish an existence and uniqueness theorem using a fixed-point argument and recent advances in hemivariational inequalities theory. Finally, we present a fully discrete finite element approximation of the model and derive error estimates for the approximate solution.

Keywords Thermo-electro-visco-elastic materials, locking piezoelectric, frictional contact problem, finite element method, error estimate

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1. Introduction

The investigation of piezoelectric materials has gained significant attention in recent years due to their extensive applications across various industries, including railways, automotive systems, civil engineering, and aeronautics. These materials possess unique properties, such as the ability to produce electrical charges under mechanical deformation and to undergo mechanical deformation when exposed to electric fields. Nonetheless, the interaction between a deformable piezoelectric structure and a conductive base presents intricate challenges. For example, friction-induced energy dissipation can heat the material, and through the pyroelectric effect, this heating affects certain piezoelectric systems by generating electrical charges or voltage. Analyzing these complex interactions requires a thorough study of coupled thermo-electro-mechanical phenomena, especially in contact problems with or without friction. Understanding these behaviors is essential for accurately modeling electro-elastic materials in real-world applications.

This study explores a contact problem involving a nonlinear thermo-visco-electro-elastic body with locking materials in contact with a rigid foundation. The model is formulated using hemivariational inequalities that account for visco-piezoelectric, thermal, subdifferential friction, and material degradation effects. For a comprehensive discussion on contact problems in piezoelectric and visco-piezoelectric materials,

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see [2, 17, 21, 24, 26–28, 32, 36, 38, 45, 46], while further insights into thermo-visco-piezoelectric materials are available in [1, 3, 7–9, 16, 18–20, 22, 29, 30, 39, 42–44, 47]. Furthermore, the concept of locking materials, first introduced by Prager [40, 41], describes materials that exhibit a significant increase in stiffness beyond a certain deformation threshold, effectively restricting further strain. This behavior is particularly observed in the study of contact problems, where interactions between a deformable body and a rigid foundation can lead to localized stress concentrations. Incorporating locking effects into our model allows for a more accurate representation of real-world materials, where excessive deformation must be restricted to prevent structural failure. In the context of thermo-electro-visco-elasticity, accounting for these effects is crucial, as they influence the overall mechanical response, frictional dissipation, and electro-thermal coupling in the system. For further discussions on the physical interpretation of locking materials, we refer the reader to [5, 21] and the references therein.

The analysis is performed over a time interval $[0, T]$, where $T > 0$, with time derivatives indicated by dots (e.g., $\dot{u} = \frac{\partial u}{\partial t}$). The focus is on thermo-electro-visco-elastic materials with locking properties, without explicitly expressing the dependence of various functions on the independent variables $x \in \Omega \cup \Gamma$. The governing laws of such materials are given as follows:

$$\sigma(t) \in \mathscr{A}\varepsilon(\dot{u}(t)) + \mathscr{B}\varepsilon(u(t)) - \mathscr{P}^T E(\varphi(t)) - \mathscr{C}\theta(t) + \partial I_{L_1}\varepsilon(u(t)), \tag{1.1}$$

$$D(t) \in \mathscr{P}\varepsilon(u(t)) + \beta E(\varphi(t)) + \mathscr{G}\theta(t) + \partial I_{L_2} E(\varphi(t)), \tag{1.2}$$

$$\dot{\theta}(t) - div\,\mathscr{K}(\nabla\theta(t)) - h_0(t) \in \mathscr{M}\varepsilon(u)(t) - \mathscr{N}E(\varphi(t)) + \partial I_{L_3}\nabla\theta(t), \tag{1.3}$$

in which σ is the stress tensor, u is the displacement field, φ represents the electric potential field and θ is the temperature field. Moreover, $\partial I_{L_1} : \mathbb{S}^d \to 2^{\mathbb{S}^d}$, $\partial I_{L_2} : L^2(\Omega) \to 2^{L^2(\Omega)}$ and $\partial I_{L_3} : L^2(\Omega) \to 2^{L^2(\Omega)}$ stand for the subdifferentials of the indicator maps of the sets L_1, L_2 and L_3, defined by

$$I_{L_i}(q, \varepsilon) = \begin{cases} 0 & \text{if} \quad \varepsilon \in L_i\,, \\ +\infty & \text{if} \quad \varepsilon \notin L_i\,. \end{cases}$$

The subsets $L_1 \subset \mathbb{S}^d$, $L_1 \subset L^2(\Omega)$ and $L_2 \subset L^2(\Omega)$ define the locking constraints and characterize the material properties. These sets can take various forms, as explored in [5]. In this paper, we specifically focus on the case of perfectly locking materials, where the sets L_1, L_2, and L_3 are given by

$$L_1 = \{\varepsilon \in \mathbb{S}^d : Q_1(\varepsilon) \le 0\}, \tag{1.4}$$

$$L_2 = \{\psi \in L^2(\Omega) : Q_2(\psi) \le 0\}, \tag{1.5}$$

and

$$L_3 = \{\theta \in L^2(\Omega) : Q_3(\theta) \le 0\}. \tag{1.6}$$

Here, the locking functions $Q_1 : \mathbb{S}^d \to \mathbb{R}$, $Q_2 : L^2(\Omega) \to \mathbb{R}$, and $Q_3 : L^2(\Omega) \to \mathbb{R}$ are convex, continuous, and the initial condition $Q_i(0) \le 0$ for $i = 1, 2, 3$.

Mathematically, models describing thermo-electro-visco-elastic materials are relatively recent advancements, as seen in works such as [20, 43]. The first contribution of this paper is the extension of these models to thermo-electro-visco-elastic contact