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Neimark-Sacker Bifurcation of a Semi-Discrete Lasota-Waźewska Model

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Abstract In this paper, we derive and analyze a semi-discrete Lasota-Waźewska model. First, the existence, uniqueness, and local dynamical properties of the positive fixed point are systematically investigated. Subsequently, we explore the existence of Neimark-Sacker bifurcation and the stability of the bifurcated invariant curve. Finally, numerical simulations are provided to illustrate the theoretical findings.

Keywords Semi-discrete Lasota-Waźewska model, Neimark-Sacker bifurcation, invariant curve, numerical simulation

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1. Introduction

In 1976, M. Waźewska-Czyźewska and A. Lasota [16] proposed the following delayed differential equation:

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -\mu N(t) + \rho e^{-\gamma N(t-h)}, \tag{1.1}$$

which is a reduced system to describe the number of red blood cells (RBCs) in an animal, where $N(t)$ represents the population of RBCs at time t, $\mu \in (0, 1)$ denotes the mortality rate for RBCs, $\rho, \gamma \in (0, +\infty)$ are constants characterizing the ability to generate new RBCs per unit time, and the delay $h > 0$ denotes the time required for RBCs to generate new cells. And system (1.1) is usually called the Lasota-Waźewska system.

The Lasota-Waźewska system and its generalized systems have been extensively studied since they were proposed. Significant progress has been made on the existence and stabilities of the positive equilibrium, positive periodic solutions, positive almost periodic solutions and positive pseudo-almost periodic solutions and so on [3–6, 8, 9, 15, 17].

These models are generally categorized into two types: continuous models defined by differential equations, and discrete models derived through discretization of continuous systems. The discrete framework is often regarded as more suitable for modeling real-world phenomena. Consequently, discretized models have garnered considerable research interest, with several discrete Lasota-Waźewska systems being proposed and analyzed in recent studies [1, 2, 7, 14]. Chen [1] discussed positive periodic solutions for a discrete Lasota-Waźewska model with impulse. Chen [2] et

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al. studied dynamic behaviors for a delay Lasota-Waźewska model with feedback control on time scales, which unified the continuous and discrete models.

In 2002, Tamas and Gabor [12] introduced a semi-discretization method for studying delayed systems, which provided a simple yet effective approach to handling delayed terms. Subsequently, this method has been applied to propose and analyze several new semi-discretized models. For instance, in [10], the authors proposed a semi-discrete hematopoiesis model and analyzed its dynamical behaviors, including the stabilities of the fixed points and the Neimark-Sacker bifurcation. Some other studies on semi-discrete models can be found in [11, 13] and related references. Additionally, Yao and Li [18] demonstrated differences in bifurcation behavior induced by distinct discretization methods within a discrete predator-prey model, highlighting the significance of methodological comparisons in discrete dynamical systems.

Inspired by the aforementioned studies, we shall investigate a semi-discrete version of system (1.1) in this paper. Our main aim is to find some new phenomena in the system. The existence, uniqueness and local dynamical behaviors of the positive fixed point are discussed. We also show that the system will undergo Neimark-Sacker bifurcation by both theoretical analysis and numerical simulations, and the stability of the bifurcated invariant curve is presented by computing the first Lyapunov coefficient. Notably, while the positive fixed point lacks an explicit analytical solution, we successfully analyze its local characteristics through the implicit function theorem and auxiliary analytical techniques.

The remainder of this paper is organized as follows. In Section 2, we derive a semi-discrete model for system (1.1), which is subsequently transformed into a discrete planar system through appropriate coordinate transformations. Section 3 presents the dynamical analysis of the proposed system, including the existence and local stability of the positive fixed point, conditions for Neimark-Sacker bifurcation occurrence, and the stability of the bifurcated invariant curve. Numerical simulations validating our theoretical results are provided in Section 4. Finally, we conclude the paper with a brief discussion in Section 5.

2. Problem and assumptions

We shall establish the semi-discrete model for system (1.1) using the method in [12] in this section. First, introduce the transformations $s = \frac{t}{h}$ and $N(t) = N(sh) = \eta(s)$, then (1.1) is changed to

$$\frac{\mathrm{d}\eta}{\mathrm{d}s} = -\delta\eta(s) + pe^{-\gamma\eta(s-1)}, \quad s \geq 0, \tag{2.1}$$

where $\delta = \mu h$, $p = \rho h$, and the delay h is turned into 1, which makes the problem simple. Assume $[s]$ is the integer that not bigger than s, and consider the semi-discrete model of (2.1):

$$\frac{\mathrm{d}\eta}{\mathrm{d}s} = -\delta\eta([s]) + pe^{-\gamma\eta([s-1])}, \quad s \neq 0, 1, 2, \cdots . \tag{2.2}$$

Obviously, equation (2.2) has piecewise constant arguments. We can directly have the following conclusion.

Lemma 2.1. *The solution $\eta(s)$ of equation (2.2) satisfies*