Journal of Nonlinear Modeling and Analysis
Volume 7, Number 5, October 2025, 1757–1767
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.1757

Double-Phase Elliptic Equations with Nonlinear Sources Existence and Uniqueness of the Weak Solution

El Mehdi Hassoune[1,†], Ahmed Jamea[2], Mohamed Laghdir[1] and Adnane Kaddiri[1]

Abstract In our work, our objective is to prove the existence and uniqueness of a weak solution to a class of nonlinear degenerate elliptic (p, q)-Laplacian problem with Dirichlet-type boundary condition by giving L^∞ data. The principal technique utilized here is the variational method alongside the theory of Orlicz spaces and Minty-Browder theorem.

Keywords Weak solution, uniqueness solutions, double phase operator, nonlinear elliptic equations

MSC(2010) 35J65, 35A02, 35J92, 35J66.

1. Introduction

The study of unbalanced integral functionals and double phase problems experienced a revolution for the Italian school under the hands of Marcellini, Mingione, Colombo, Baroni, et al, their study and remark is based entirely on the work of Zikhov in order to describe the behavior of phenomena arising in nonlinear elasticity. In reality, Zhikov aimed to offer prototypes for highly anisotropic materials within the framework of homogenization. Specifically, Zhikov explored three distinct functional models in connection with the Lavrentiev phenomenon. These models are:

$$\begin{aligned}\mathcal{M}(u) &:= \int_\Omega c(x)|\nabla u|^2 dx,\\ \mathcal{N}(u) &:= \int_\Omega |\nabla u|^{p(x)} dx,\\ \mathcal{T}_{p,q}(u) &:= \int_\Omega c(x)|u|^p dx + a(x)|\nabla u|^q dx,\end{aligned} \tag{1.1}$$

where $0 < 1/c(.) \in L^1(\Omega), 1 < p(.) < \infty, 1 < p < q, 0 \le a(.) \le L$. The functional $\mathcal{M}$ has been extensively investigated in the context of equations that incorporate

†the corresponding author.
Email address: mahdihassoune@gmail.com(E. Hassoune), a.jamea77@gmail.com(A. Jamea), laghdirm@gmail.com(M. Laghdir), adnanekaddiri@gmail.com(A. Kaddiri)
[1]Laboratoire LITE, Faculté des Sciences, Université Chouaib Doukkali El Jadida, Morocco
[2]CRMEF Casablanca-Settat, S.P. El Jadida,El Jadida, Morocco

Muckenhoupt weights. The functional $\mathcal{N}$ has recently garnered significant attention, leading to a substantial body of literature devoted to its study. The functional $\mathcal{T}_{p,q}$ in (1.1) is presented as an enhanced form of $\mathcal{N}$. Here as well, the coefficient $a(.)$ influences the geometry of the composite consisting of two differential materials with hardening exponents p and q, respectively. The functionals depicted in (1.1) belong to the category of functionals with nonstandard growth conditions of type (p,q), as classified by Marcellini.

Let $\Omega \subset \mathbb{R}^N$, $(N \geq 2)$ be an open bounded domain and let $p,q \in (1,\infty)$. Our aim in this work is to study the existence and uniqueness of the weak solution to the following nonlinear degenerate elliptic problem

$$\begin{cases} -div\left(\mathcal{A}_{p,q}(x,\nabla u,\theta(u))\right) + g(u) = f \ \ in \ \ \Omega \\ u = 0 \ \ on \ \ \partial\Omega, \end{cases} \tag{1.2}$$

where $\mathcal{A}_{p,q}(x,\nabla u,\theta(u)) = |\nabla u - \theta(u)|^{p-2}(\nabla u - \theta(u)) + a(x)|\nabla u - \theta(u)|^{q-2}(\nabla u - \theta(u))$ and $g(u) = |u|^{p-2}u$ and θ are continuous function defined from $\mathbb{R}$ to $\mathbb{R}^N$, the datum f is in L^∞ and $a : \Omega \to \mathbb{R}$ is an Lipschitz continuous map, $a(x) \geq 0$ for all $x \in \overline{\Omega}$. These types of PDEs model several physical phenomena, including elastic mechanics, electrorheological fluid dynamics, and image processing, among others, for example.

Reaction-Diffusion Systems are mathematical models used to describe the spatial and temporal evolution of multiple substances or populations under the influence of two key processes: reaction and diffusion. These systems have widespread applications across various fields such as chemistry, biology, ecology, and physics, due to their ability to model phenomena involving pattern formation, wave propagation, and concentration dynamics.

Reaction refers to the interactions between substances (e.g., chemical reactions or biological processes like predator-prey dynamics), where the concentration of one or more substances changes over time due to chemical or biological reactions.

Diffusion describes the spreading or movement of substances from regions of high concentration to low concentration, driven by concentration gradients.

A general mathematical model for a Reaction-Diffusion System is described by a system of partial differential equations (PDEs).

$$-div(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) + R(u) = 0$$

where a(x) is the modulating coefficient that switches between p and q-phase behaviors, and R is the reaction term.

This paper primarily focuses on the study of unbalanced double-phase problems. We recall some basic properties of the Musielak-Orlicz and Sobolev spaces and introduce a new non-homogeneous differential operator, which will be utilized in this work. Using the Minty-Browder theorem and suitable assumptions, we prove the existence of a weak solution to our problem. Additionally, we employ fundamental lemmas to establish the uniqueness of the solution.

Recently, when $a(x) \equiv 0$, the existence and uniqueness of entropy solutions for the p-Laplace equation were proved by A. Sabri and A. Jamea. When $\theta \equiv 0$, the study of existence and uniqueness of entropy solutions for the problem has been further investigated. Moreover, R. Arora and S. Shmarev proved the existence and uniqueness of strong solutions when $\theta \equiv 0$, and for p non constant and A. Sabri,