Journal of Nonlinear Modeling and Analysis
Volume 7, Number 5, October 2025, 1792–1810
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.1792

On a Class of Eigenvalue Elliptic Systems in Fractional Sobolev Spaces with Variable Exponents

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Abstract In this work, we consider a class of eigenvalue elliptic systems involving the fractional $(p(x,.), q(x,.))$-Laplacian operators. Our main tools are based on Mountain Pass Theorem and Fountain Theorem.

Keywords Elliptic systems, generalized fractional Sobolev spaces, variational methods, variational principle, fractional $p(x,.)$-Laplacian, mountain pass Theorem, Fountain Theorem

MSC(2010) 35J48, 35J50, 46E35.

1. Introduction

The birth of fractional calculus was in a letter dated in 1695 when L'Hôpital wrote to Leibniz and asked him about the nth-derivative of the function $f(x) = x$, $\frac{D^n x}{Dx^n}$. What could be the result if $n = \frac{1}{2}$? Leibniz responded: "An apparent paradox from which, one day, useful consequences will be drawn."

After this first discussion between L'Hôpital and Leibniz, fractional calculus became above all for big mathematicians and can be traced back to L'Hôpital (1695), Wallis (1697), Euler (1738), Laplace (1812), Lacroix (1820), Fourier (1822), Abel (1823), Liouville (1832), Riemann (1847), Leibniz (1853), Grunwald (1867), Letnikov (1868) and many others. We refer the reader to [17,27,30] and the references therein for a detailed exposition about the history of the classical fractional calculus.

Thanks to these classical definitions, fractional calculus becomes a venerable branch of mathematics in the last century, and fractional operators give more development to the fields of Potential theory, Probability, Hyper singular integrals, Harmonic analysis, Functional analysis, Pseudo-differential operators, Semigroup theory etc. In particular, the fractional Laplacian operator $(-\Delta)^s$, $s \in (0,1)$ is a pseudo-differential operator which has various definitions in different fields: Fourier transform, distributional definition, Bochner's definition, Balakrishnan's definition,

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singular integral definition, quadratic definition, semigroup definition, definition as harmonic extension, and definition as the inverse od Riez potential. We refer to Kwasnicki [21] who collected all these definitions and established the equivalence between them.

In the last decade, great attention has been given to the study of nonlinear nonlocal case. More precisely, the problems involving the fractional p-Laplacian operator $(-\Delta)_p^s$. We refer to Di Nezza et al. [26] for a comprehensive introduction to the study of nonlocal problems. Basic properties, embedding theorems and regularity results are established. The paper opens the door for many mathematicians to deal with general problems in the field of partial differential equations. In the context of nonhomogeneous materials (such as electrorheological fluids and smart fluids), the use of Lebesgue and Sobolev spaces L^p and $W^{s,p}$ seems to be inadequate, which leads to the study of variable exponent functional spaces. The study of problems which involves the $p(.)$-Laplacian and the corresponding elliptic equations constitutes promising a domain of research. The interest in studying such problems was stimulated by their applications in many physical phenomena such as conservation laws, ultra-materials and water waves, optimization, population dynamics, soft thin films, mathematical finance, phases transitions, stratified materials, anomalous diffusion, crystal dislocation, semipermeable membranes, flames propagation, ultra-relativistic limits of quantum mechanics, electrorheological fluids. We refer the reader to [7–9, 13, 26, 28, 29, 33] for details.

Now, what results can be recovered if the $p(.)$-Laplace operator is replaced by the fractional $p(x,.)$-Laplacian of the form $(-\Delta)^s_{p(x,.)}$? In 2017, Kaufmann et al. in [19] introduced the fractional Sobolev spaces with variable exponent $W^{s,q(x),p(x,y)}(\Omega)$. They established continuous and compact embedding theorems of these spaces into variable exponent Lebesgue spaces with the restriction $p(x,x) < q(x)$, and as applications, they also proved an existence result for nonlocal problems involving the fractional $p(x,y)$-Laplacian. In [6], Bahrouni et al. presented some further qualitative properties of both on this function space and the related $p(x,.)$-Laplacian operator $\mathcal{L}_{p(x,.)}$.

Under the restricted condition $p(x,x) < q(x)$, the space $W^{s,q(x),p(x,y)}(\Omega)$ is in fact not a generalization of the typical fractional Sobolev space $W^{s,p}(\Omega)$ with a constant exponent. However, Ho et al. [18] and Azroul et al. [2] provided some fundamental embeddings for the fractional Sobolev space with variable exponent to cover the case $p(x,x) = q(x)$ and their applications such as a priori bounds and multiplicity of solutions of the fractional $p(x,.)$-Laplacian problems. We also refer to [12] in which the authors proved a trace theorem in fractional Sobolev spaces with variable exponents. Indeed, fractional Sobolev spaces with variable exponents have been studied in depth during the last decade. We refer the interested reader to [2, 5, 6, 10, 22, 23, 25] and the references therein for some recent existence results for fractional type problems driven by a $p(x,.)$-Laplacian operator.

The purpose of this paper is to study the following fractional elliptic eigenvalue system