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Existence, Asymptotics and Computation of Solutions of Nonlinear Sturm-Liouville Problems

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Abstract This paper deals with the existence, asymptotics and computa-tion of solutions of nonlinear Sturm-Liouville problems with general separated boundary conditions. The approach centers first on converting these prob-lems into Hammerstein integral equations with modified argument, and then applying the Banach and Rothe fixed point theorems to solve them. This ap-proach not only enabled us to prove existence theorems for these problems, but also to derive general and accurate asymptotic formulae for their solutions. Finally, an illustrative numerical example is presented using the Picard's iter-ation method.

Keywords Approximate methods, asymptotic expansion, Jacobi elliptic functions, ordinary differential equation, nonlinear eigenvalue problem

MSC(2010) 34B15, 34B24, 34L16, 34L20, 47H30.

1. Introduction

Consider the nonlinear Sturm-Liouville equation

$$-u''(x) = \lambda u(x) - f(x, u(x)),\ x \in I := [a, b], \tag{1.1}$$

defined on the compact interval I and subject to the separated boundary conditions

$$a_1u(a) + a_2u'(a) = 0,\ |a_1| + |a_2| \neq 0,\ a_1, a_2 \in \mathbb{C}, \tag{1.2}$$
$$b_1u(b) + b_2u'(b) = 0,\ |b_1| + |b_2| \neq 0,\ b_1, b_2 \in \mathbb{C}. \tag{1.3}$$

Second-order nonlinear Sturm-Liouville equations are important due to their numerous real-world applications. The simple pendulum is a typical example, which is governed by the nonlinear equation

$$y'' + k^2 \sin(y) = 0, \tag{1.4}$$

where the constant $k \neq 0$ depends on the length of the pendulum and on gravity. Note that, by using the transformation $u := \sin\left(\frac{1}{2}y\right)$, one can reduce equation (1.4) to an equation of the form

$$-u'' = \lambda u - Au^3, \tag{1.5}$$

which is a special case of Sturm-Liouville equation (1.1) and is integrated by elliptic functions. For more details, see [12]. Equipped with boundary conditions of the

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form (1.3), a system of equations (1.1) and (1.3) is called a (nonlinear) Sturm-Liouville problem, which also occurs very frequently in applied mathematics. For details related to the Sturm-Liouville theory, the reader is referred to the books [18, 38]. These nonlinear problems have been studied for different purposes and under diverse forms by many authors. For a study of the existence of solutions or positive solutions see, for example, [2, 5, 8, 10, 16, 23, 24, 26, 30]. For a study of the asymptotic behavior of the solutions see, for example, [4, 6, 15, 22, 31, 32]. However, limited numerical studies have been carried out on them. The reader may see, for example, [1, 29, 34, 37, 39]. Note that some problems in the previously mentioned articles are special cases of problem (1.1)–(1.3).

In a recent article [17], we studied the existence and asymptotics of solutions of problem (1.1)–(1.3) when the nonlinear term f has the form Qu, with $Q \in L^1(I)$, using the homotopy perturbation method. We also have presented several numerical examples illustrating our theoretical results. In this paper, the objective is the same but the approach is different: we investigate the solvability of problem (1.1)–(1.3), as well as the asymptotic behavior of the pair (λ, u) as $|\lambda| \to \infty$, when the function f is nonlinear. We also present an iterative scheme that computes the solutions of problem (1.1)–(1.3). The approach adopted consists first in converting problem (1.1)–(1.3) into a nonlinear integral equation of Hammerstein type with modified argument, and then applying the Banach and Rothe fixed point theorems to solve it. This conversion is of interest for two primary reasons: it is better suited for proving the existence of solutions, and it enables the analysis of their qualitative properties. Moreover, Hammerstein integral equations have been intensively investigated in the literature, using many different approaches and methods, from both theoretical and numerical viewpoints, see, for example, [3, 7, 9, 11, 13, 21, 27]; see also [14, 25, 35] for integral equations with modified argument. For more details on the theory of integral equations, we refer the reader to the books [19, 20].

The outline of the paper is as follows: In Section 2, we state some preliminaries and notations. In Section 3, we begin by solving the integral equation in question using Rothe and Banach fixed point theorems. Then, we state and prove our existence theorems. As a direct consequence of these results, we show that if the boundary condition constants a_i and b_i, $i = 1, 2$, are real and f is continuous on $I \times V$, where V is some compact neighborhood of 0, then problem (1.1)–(1.3) has infinitely many solutions, which are twice continuously differentiable on I. Finally, we derive general and accurate asymptotic formulae for the solutions of problem (1.1)–(1.3) for sufficiently large $|\lambda|$. In the last section, we consider equation (1.5) with Dirichlet boundary conditions in order to illustrate our main results.

2. Preliminaries and notations

Some basic definitions and results are provided below (see, for example, [28, 33, 36]), which will be used in the next section.

Definition 2.1. Let $(\mathcal{X}, d)$ be a compact metric space. We say that $\mathcal{A} \subset C(\mathcal{X}, \mathbb{C})$ is uniformly bounded if there exists $M > 0$ such that $|g(x)| \leq M$, $\forall g \in \mathcal{A}$, $\forall x \in \mathcal{X}$.

Definition 2.2. Let $(\mathcal{X}, d)$ be a compact metric space. We say that $\mathcal{A} \subset C(\mathcal{X}, \mathbb{C})$ is equicontinuous if for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for all $x, y \in \mathcal{X}$ with $d(x, y) < \delta_\varepsilon$ it follows that $|g(x) - g(y)| < \varepsilon$, $\forall g \in \mathcal{A}$.