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On a Class of Nonlinear Elliptic Problems Involving the $\alpha(z)$-Biharmonic Operator with an $l(z)$-Hardy Term

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Abstract By applying the Mountain Pass Theorem, we establish the existence of a weak solution for a class of nonlinear elliptic problem involving an $\alpha(z)$-biharmonic operator and with an $l(z)$-hardy term in a bounded domain of $\mathbb{R}^N$. Provided that certain additional assumptions are made regarding the nonlinearities, the corresponding functional will satisfy the Palais-Smale condition.

Keywords $\alpha(z)$-biharmonic operator, variable exponents, $l(z)$-Hardy term, Hardy-Rellich inequality, Mountain Pass Theorem

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1. Introduction

In this paper, we investigate the existence of weak solution of the following elliptic problem involving an $\alpha(z)$-biharmonic operator and $l(z)$-hardy term

$$\begin{cases} \Delta^2_{\alpha(z)}\upsilon = \lambda_1 \dfrac{|\upsilon|^{l(z)-2}\upsilon}{\gamma(z)^{2l(z)}} + \lambda_2 Q(z)|\upsilon|^{\beta(z)-2}\upsilon + \lambda_3 g(z,\upsilon) & in \quad \mathfrak{D}, \\ \\ \Delta\upsilon = \upsilon = 0 & on \quad \partial\mathfrak{D}, \end{cases} \tag{1.1}$$

where $\mathfrak{D}$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary. We indicate by $\gamma(z) := dist(z,\partial\mathfrak{D})$ the distance from the point $z \in \mathfrak{D}$ to the boundary $\partial\mathfrak{D}$, $\Delta^2_{\alpha(z)}\upsilon = \Delta\left(|\Delta\upsilon|^{\alpha(z)-2}\Delta\upsilon\right)$ is the $\alpha(z)$-biharmonic operator, the exponents α, β and l are continuous functions on $\overline{\mathfrak{D}}$, λ_1, λ_2, λ_3 are three positive parameters, $g : \mathfrak{D}\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function and Q is an indefinite weight function.

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Nonlinear singular elliptic problems have been a popular topic of study in recent years. They arise in some parts of science, such as boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, nonlinear electrorheological fluids and the flow in porous media. This has led to a great deal of excitement and interest from a number of authors in recent years, as the investigation of the existence and multiplicity of solutions for problems involving biharmonic, α-biharmonic and $\alpha(z)$-biharmonic operators, where α is a continuous function, has attracted significant interest (see [2, 4, 13–15, 18, 20, 23–25]).

The same problem, for $\lambda_2 = \lambda_3 = 0$ is studied by Laghzal and Touzani [18]. The authors determined that there is at least one non-decreasing sequence of non-negative eigenvalues for their problem.

In [1], Taarabti, El Allali and Hadddouch studied the existence of solutions to a nonhomogeneous eigenvalue problem with $\lambda_1 = \lambda_2 = 0$, by considering different situations with respect to the growth and they proved that a continuous family of eigenvalues exists.

In [16], the present author studied the existence of the following fourth-order, nonlinear elliptic problem

$$\begin{cases} \Delta^2_{\alpha(z)}\upsilon + a(z)|\upsilon|^{\alpha(z)-2}\upsilon = \lambda f(z,\upsilon) & \text{in } \mathfrak{D}, \\ \\ \upsilon = \Delta\upsilon = 0 & \text{on } \partial\mathfrak{D}, \end{cases}$$

for $\lambda > 0$, by using the Mountain Pass Theorem.

The remaining sections are organised as follows. In Section 2, we present fundamental results for the generalized Lebesgue-Sobolev $L^{\alpha(z)}(\mathfrak{D})$ and $W^{m,\alpha(z)}(\mathfrak{D})$. Moreover, the Mountain Pass Theorem is recalled (Theorem 2.2). Section 3, we prove that weak solutions exist for (1.1) by presenting several lemmas.

2. Preliminaries

For the reader's convenience, we recall in what follows some necessary background knowledge and propositions concerning the generalized Lebesgue-Sobolev spaces $L^{\alpha(z)}(\mathfrak{D})$ and $W^{m,\alpha(z)}(\mathfrak{D})$ where $\mathfrak{D}$ is an open subset of $\mathbb{R}^{\mathbb{N}}$(see for example [3, 6, 7, 9, 11, 12, 17, 19, 22]).

Let

$$C_+(\overline{\mathfrak{D}}) = \{\alpha \in C(\overline{\mathfrak{D}}) : \alpha(z) > 1, \text{ for every } z \in \overline{\mathfrak{D}}\}.$$

For every $\alpha \in C_+(\overline{\mathfrak{D}})$, we define

$$\alpha^+ = \max\{\alpha(z);\ z \in \overline{\mathfrak{D}}\} \textit{ and } \alpha^- = \min\{\alpha(z);\ z \in \overline{\mathfrak{D}}\}.$$

The generalized Lebesgue space $L^{\alpha(z)}(\mathfrak{D})$ is defined as

$$L^{\alpha(z)}(\mathfrak{D}) = \left\{\upsilon : \mathfrak{D} \to \mathbb{R},\ \textit{measurable and } \int_{\mathfrak{D}} |\upsilon(z)|^{\alpha(z)} dz < \infty\right\}.$$

We endow it with the Luxemburg norm

$$\|\upsilon\|_{\alpha(z)} = \inf\left\{\theta > 0 : \int_{\mathfrak{D}} \left|\frac{\upsilon(z)}{\theta}\right|^{\alpha(z)} dz \le 1\right\}.$$