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Existence, Uniqueness and Fourth-Order Numerical Method for Solving Fully Third-Order Nonlinear ODE with Integral Boundary Conditions*

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Abstract In this paper, we establish the existence, uniqueness and construct fourth-order numerical method for solving fully third-order nonlinear differential equation with integral boundary conditions. The method is based on the discretization of an iterative method on continuous level with the use of the trapezoidal quadrature formulas with corrections. Some examples demonstrate the applicability of the theoretical results of existence and uniqueness of solution and the fourth-order convergence of the proposed numerical method. The approach used for the third-order nonlinear differential equation with integral boundary conditions can be applied to differential equations of any order.

Keywords Third-order nonlinear differential equation, integral boundary conditions, iterative method, fourth-order convergence

MSC(2010) 34B15, 34L30, 65L10, 65L12.

1. Introduction

Third-order differential equations arise in many different fields of mechanics and physics, for example, in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [9]. Recently, third-order two-point or multipoint boundary value problems (BVPs) have attracted a lot of attention (see e.g., [1, 2, 4, 8, 14, 19] and references therein). It is known that BVPs with integral boundary conditions cover multipoint BVPs as special cases. It is worth mentioning here some works on third-order BVPs with integral boundary conditions [3, 10–12, 21–23].

In this paper we consider the following boundary value problem (BVP)

$$u'''(t) = f(t, u(t), u'(t), u''(t)),\ t \in (0, 1), \tag{1.1}$$

$$u(0) = c_1, u''(0) = c_2, u(1) = \int_0^1 g(s)u(s)ds + c_3, \tag{1.2}$$

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where f, g are continuous functions and c_1, c_2, c_3 are real numbers. Some simplified versions of the above problem were studied in [11, 12, 22]. Namely, in [22] Zhao et al. investigated the existence, nonexistence, and multiplicity of positive solutions for the problem

$$u'''(t) = f(t, u(t)),\ t \in (0,1), \tag{1.3}$$

$$u(0) = 0, u''(0) = 0, u(1) = \int_0^1 g(s)u(s)ds \tag{1.4}$$

in ordered Banach spaces by means of fixed-point principle in cone and the fixed-point index theory for strict set contraction operator. One example was given to illustrate the existence of positive solutions although this solution was not shown. Later, in 2012 and 2013 Guo et al. [11, 12] by using the fixed-point index theory in a cone and nonlocal Green function obtained the existence of at least one positive solution for the problem of the type (1.3)-(1.4), where $f = f(t, u, u'')$ and $f = f(t, u, u')$, respectively. In [12] a complicated example was designed to satisfy the sufficient conditions of the existence.

Remark that except for boundary conditions (1.2), other types of integral boundary conditions for the fully or not fully third-order differential equations have also attracted attention from many authors. Among them there are the boundary conditions

$$u(0) = 0, u'(0) = 0, u(1) = \int_0^1 g(s)u(s)ds. \tag{1.5}$$

Some authors, e.g., Guendouz et al. [10] studied the existence of positive solutions of the BVPs (1.1), (1.5); Smirnov [17] investigated the existence and uniqueness of solutions by using the Green function of the differential equation with nonlocal boundary conditions. A year after, Smirnov studied the existence of multiple positive solutions of the equation (1.3) with the boundary conditions

$$u(0) = 0, u'(0) = 0, u(1) = \lambda[u], \tag{1.6}$$

where $\lambda[u] = \int_0^1 u(s)d\Lambda(s)ds$ is a linear functional on $C[0,1]$ given by Stieltjes integral with $\hat{\imath}$ a suitable function of bounded variation. Boundary conditions (1.6) include as special cases multipoint conditions and integral conditions. In [21] Zhang and Sun investigated the existence of monotone positive solutions for the following nonlocal problem

$$\begin{aligned}
&u''' + f(t, u, u') = 0, \quad t \in (0,1),\\
&u(0) = 0,\\
&au'(0) - bu''(0) = \alpha[u],\\
&cu'(1) + du''(1) = \beta[u],
\end{aligned}$$

where $\alpha[u] = \int_0^1 u(s)dA(s)ds, \beta[u] = \int_0^1 u(s)dB(s)ds$ are linear functionals on $C^1[0,1]$ given by Riemann-Stieltjes integrals. Very recently, Szajnowska and Zim [23] studied the existence of positive solutions to the third-order differential equation of the form

$$-u''' + m^2u' = f(t, u, u'),$$

subject to the non-local boundary conditions

$$u(0) = 0, u'(0) = \alpha[u], u'(1) = \beta[u],$$