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A New Approach to Hermite-Hadamard-Type Inequality with Proportional Caputo-Hybrid Operator

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Abstract Fractional calculus plays a crucial role in mathematics and applied sciences as it extends classical analysis, overcoming many of its constraints. Moreover, using the innovative hybrid fractional operator, which merges the proportional and Caputo operators, is beneficial in numerous domains of computer science and mathematics. In this study, we focus on the proportional Caputo-hybrid operator due to its wide range of applications. Firstly, we present a new extension of Hermite-Hadamard-type inequalities for the proportional Caputo-hybrid operator and derive an identity. Then, utilizing this novel generalized identity, we establish significant integral inequalities associated with the right-hand side of Hermite-Hadamard-type inequalities for the proportional Caputo-hybrid operator. Furthermore, we provide illustrative examples accompanied by the graphs to demonstrate the newly established inequalities.

Keywords Hermite-Hadamard-type inequalities, trapezoid-type inequalities, convex functions, proportional Caputo-Hybrid operator

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1. Introduction

Convex analysis is a branch of mathematics that has been applied to various fields, including optimization theory, engineering applications, and physics. It plays a crucial role, especially in the field of inequalities, in various areas of mathematics. Charles Hermite and Jacques Hadamard [14], [17] separately studied the Hermite-Hadamard inequality, which is one of the most well-known inequalities in convex theory. The following is an expression for this inequality:

$$\Lambda\left(\frac{\psi+\varphi}{2}\right) \leq \frac{1}{\varphi-\psi}\int_{\psi}^{\varphi}\Lambda(\pi)d\pi \leq \frac{\Lambda(\psi)+\Lambda(\varphi)}{2}, \tag{1.1}$$

where $\Lambda : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\psi, \varphi \in I$ with $\psi < \varphi$. If Λ is concave, then both inequalities in the statement hold in the reverse direction. A convex function's average value on a compact interval can

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be found using the Hermite-Hadamard inequality, which provides both upper and lower bounds. Numerous disciplines use this inequality, such as integral calculus, probability theory, statistics, and number theory. The Hermite-Hadamard inequality is widely studied and applied in various mathematical fields. Their applications keep growing as new problems come up, which makes them an invaluable tool for resolving a variety of mathematical issues. Also, researchers have devoted considerable attention to studying the Hermite-Hadamard inequality, particularly its associations with the trapezoidal and midpoint inequalities. Dragomir and Agarwal [13] initially established trapezoid-type inequalities for the condition of convex functions, while Kırmacı [25] first proved midpoint-type inequalities for the condition of convex functions. Since these inequalities first appeared, there has been a lot of work in this field [1, 6, 19].

There is a strong historical basis for fractional calculus. Using fractional calculus, we may more precisely characterize the dynamics of complex systems, particularly those displaying non-integer order dynamics. It broadens the scope of traditional calculus by incorporating fractional orders. It has gained more importance and has found applications in various fields of science and engineering. The study of fractional calculus has gained popularity recently due to the development of new fractional integral and derivative concepts such as Caputo-Fabrizio [11], Atangana-Baleanu [3], tempered [31], etc. These concepts are crucial in understanding the dynamics of intricate systems in a variety of scientific and engineering fields.

Riemann-Liouville integral operators, one of the basic fractional integral operators, are defined as follows [24]:

Definition 1.1. For $\Lambda \in L_1[\psi, \varphi]$, the Riemann-Liouville integrals of order $\mathfrak{s} > 0$ are given by

$$J^{\mathfrak{s}}_{\psi+}\Lambda(x) = \frac{1}{\Gamma(\mathfrak{s})}\int_{\psi}^{x} (x-\mathfrak{m})^{\mathfrak{s}-1}\Lambda(\mathfrak{m})d\mathfrak{m}, \quad x > \psi$$

and

$$J^{\mathfrak{s}}_{\varphi-}\Lambda(x) = \frac{1}{\Gamma(\mathfrak{s})}\int_{x}^{\varphi} (\mathfrak{m}-x)^{\mathfrak{s}-1}\Lambda(\mathfrak{m})d\mathfrak{m}, \quad x < \varphi.$$

Here, $\Gamma(\mathfrak{s})$ is the Gamma function and $J^{0}_{\psi+}\Lambda(\pi) = J^{0}_{\varphi-}\Lambda(\pi) = \Lambda(\pi)$. Obviously, the Riemann-Liouville integrals will be equal to classical integrals for the condition $\mathfrak{s} = 1$.

In [36], Sarıkaya and Yıldırım introduced an alternative expression of the Hermite-Hadamard inequality using fractional integrals:

Theorem 1.1. *Let $\Lambda : [\psi, \varphi] \to \mathbb{R}$ be a function with $0 \le \psi < \varphi$ and $\Lambda \in L_1[\psi, \varphi]$. If Λ is a convex function on $[\psi, \varphi]$, then the following inequalities for fractional integrals hold:*

$$\Lambda\left(\frac{\psi+\varphi}{2}\right) \le \frac{\Gamma(\mathfrak{s}+1)}{2(\varphi-\psi)^{\mathfrak{s}}}[J^{\mathfrak{s}}_{(\frac{\psi+\varphi}{2})+}\Lambda(\varphi) + J^{\mathfrak{s}}_{(\frac{\psi+\varphi}{2})-}\Lambda(\psi)] \le \frac{\Lambda(\psi)+\Lambda(\varphi)}{2}$$

with $\mathfrak{s} > 0$.

Afterwards, Sarıkaya et al. [35] and Iqbal et al. [21] developed numerous inequalities of the fractional trapezoid-type inequalities and the midpoint-type inequalities for the convex functions, respectively. Further reading on fractional integral inequalities can be found in [5, 8–10, 23, 26, 33], along with the references mentioned there.