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A New Approach to Hermite-Hadamard-Type Inequality with Proportional Caputo-Hybrid Operator

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Abstract Fractional calculus plays a crucial role in mathematics and applied sciences as it extends classical analysis, overcoming many of its constraints. Moreover, using the innovative hybrid fractional operator, which merges the proportional and Caputo operators, is beneficial in numerous domains of computer science and mathematics. In this study, we focus on the proportional Caputo-hybrid operator due to its wide range of applications. Firstly, we present a new extension of Hermite-Hadamard-type inequalities for the proportional Caputo-hybrid operator and derive an identity. Then, utilizing this novel generalized identity, we establish significant integral inequalities associated with the right-hand side of Hermite-Hadamard-type inequalities for the proportional Caputo-hybrid operator. Furthermore, we provide illustrative examples accompanied by the graphs to demonstrate the newly established inequalities.

Keywords Hermite-Hadamard-type inequalities, trapezoid-type inequalities, convex functions, proportional Caputo-Hybrid operator

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1. Introduction

Convex analysis is a branch of mathematics that has been applied to various fields, including optimization theory, engineering applications, and physics. It plays a crucial role, especially in the field of inequalities, in various areas of mathematics. Charles Hermite and Jacques Hadamard [14], [17] separately studied the Hermite-Hadamard inequality, which is one of the most well-known inequalities in convex theory. The following is an expression for this inequality:

$$\Lambda\left(\frac{\psi+\varphi}{2}\right) \leq \frac{1}{\varphi-\psi}\int_{\psi}^{\varphi} \Lambda(\pi)d\pi \leq \frac{\Lambda(\psi)+\Lambda(\varphi)}{2}, \tag{1.1}$$

where $\Lambda : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\psi, \varphi \in I$ with $\psi < \varphi$. If Λ is concave, then both inequalities in the statement hold in the reverse direction. A convex function's average value on a compact interval can

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be found using the Hermite-Hadamard inequality, which provides both upper and lower bounds. Numerous disciplines use this inequality, such as integral calculus, probability theory, statistics, and number theory. The Hermite-Hadamard inequality is widely studied and applied in various mathematical fields. Their applications keep growing as new problems come up, which makes them an invaluable tool for resolving a variety of mathematical issues. Also, researchers have devoted considerable attention to studying the Hermite-Hadamard inequality, particularly its associations with the trapezoidal and midpoint inequalities. Dragomir and Agarwal [13] initially established trapezoid-type inequalities for the condition of convex functions, while Kırmacı [25] first proved midpoint-type inequalities for the condition of convex functions. Since these inequalities first appeared, there has been a lot of work in this field [1,6,19].

There is a strong historical basis for fractional calculus. Using fractional calculus, we may more precisely characterize the dynamics of complex systems, particularly those displaying non-integer order dynamics. It broadens the scope of traditional calculus by incorporating fractional orders. It has gained more importance and has found applications in various fields of science and engineering. The study of fractional calculus has gained popularity recently due to the development of new fractional integral and derivative concepts such as Caputo-Fabrizio [11], Atangana-Baleanu [3], tempered [31], etc. These concepts are crucial in understanding the dynamics of intricate systems in a variety of scientific and engineering fields.

Riemann-Liouville integral operators, one of the basic fractional integral operators, are defined as follows [24]:

Definition 1.1. For $\Lambda \in L_1[\psi, \varphi]$, the Riemann-Liouville integrals of order $\mathfrak{s} > 0$ are given by

$$J_{\psi+}^{\mathfrak{s}}\Lambda(x) = \frac{1}{\Gamma(\mathfrak{s})}\int_{\psi}^{x} (x-\mathfrak{m})^{\mathfrak{s}-1}\Lambda(\mathfrak{m})d\mathfrak{m}, \quad x > \psi$$

and

$$J_{\varphi-}^{\mathfrak{s}}\Lambda(x) = \frac{1}{\Gamma(\mathfrak{s})}\int_{x}^{\varphi} (\mathfrak{m}-x)^{\mathfrak{s}-1}\Lambda(\mathfrak{m})d\mathfrak{m}, \quad x < \varphi.$$

Here, $\Gamma(\mathfrak{s})$ is the Gamma function and $J_{\psi+}^{0}\Lambda(\pi) = J_{\varphi-}^{0}\Lambda(\pi) = \Lambda(\pi)$. Obviously, the Riemann-Liouville integrals will be equal to classical integrals for the condition $\mathfrak{s} = 1$.

In [36], Sarıkaya and Yıldırım introduced an alternative expression of the Hermite-Hadamard inequality using fractional integrals:

Theorem 1.1. *Let $\Lambda : [\psi, \varphi] \to \mathbb{R}$ be a function with $0 \le \psi < \varphi$ and $\Lambda \in L_1[\psi, \varphi]$. If Λ is a convex function on $[\psi, \varphi]$, then the following inequalities for fractional integrals hold:*

$$\Lambda\left(\frac{\psi+\varphi}{2}\right) \le \frac{\Gamma(\mathfrak{s}+1)}{2(\varphi-\psi)^{\mathfrak{s}}}[J_{\left(\frac{\psi+\varphi}{2}\right)+}^{\mathfrak{s}}\Lambda(\varphi) + J_{\left(\frac{\psi+\varphi}{2}\right)-}^{\mathfrak{s}}\Lambda(\psi)] \le \frac{\Lambda(\psi)+\Lambda(\varphi)}{2}$$

with $\mathfrak{s} > 0$.

Afterwards, Sarıkaya et al. [35] and Iqbal et al. [21] developed numerous inequalities of the fractional trapezoid-type inequalities and the midpoint-type inequalities for the convex functions, respectively. Further reading on fractional integral inequalities can be found in [5, 8–10, 23, 26, 33], along with the references mentioned there.

The following definition is also important for fractional analysis [32]:

Definition 1.2. Let $\mathfrak{s} > 0$ and $\mathfrak{s} \notin \{1, 2, ...\}$, $n = [\mathfrak{s}] + 1$, $\Lambda \in AC^n[\psi, \varphi]$, the space of functions having $n - th$ derivatives absolutely continuous. The left-sided and right-sided Caputo fractional derivatives of order $\mathfrak{s}$ are defined as follows:

$$^{C}D^{\mathfrak{s}}_{\psi+}\Lambda(x) = \frac{1}{\Gamma(n-\mathfrak{s})}\int_{\psi}^{x}(x-\mathfrak{m})^{n-\mathfrak{s}-1}\Lambda^{(n)}(\mathfrak{m})d\mathfrak{m}, \quad x > \psi$$

and

$$^{C}D^{\mathfrak{s}}_{\varphi^-}\Lambda(x) = \frac{1}{\Gamma(n-\mathfrak{s})}\int_{x}^{\varphi}(\mathfrak{m}-x)^{n-\mathfrak{s}-1}\Lambda^{(n)}(\mathfrak{m})d\mathfrak{m}, \quad x < \varphi.$$

If $\mathfrak{s} = n \in \{1, 2, 3, ...\}$ and the usual derivative $\Lambda^{(n)}(x)$ of order n exists, then Caputo fractional derivative $^{C}D^{\mathfrak{s}}_{\psi+}\Lambda(x)$ coincides with $\Lambda^{(n)}(x)$ whereas $^{C}D^{\mathfrak{s}}_{\varphi^-}\Lambda(x)$ with exactness to a constant multiplier $(-1)^n$. For $n = 1$ and $\mathfrak{s} = 0$, we have $^{C}D^{\mathfrak{s}}_{\psi+}\Lambda(x)$ $= {^{C}D^{\mathfrak{s}}_{\varphi^-}}\Lambda(x) = \Lambda(x)$.

The Riemann-Liouville fractional derivative is generated by differentiating the fractional integral of a function with respect to its independent variable of order n, whereas the Caputo derivative is characterized as the application of a fractional integral to a standard derivative of the function. Compared to the traditional Riemann-Liouville fractional derivative when fractional differential equations are taken into account, the Caputo fractional derivative requires more appropriate initial conditions [12]. Besides, the operator of proportional derivative denoted as $^{P}D_{\mathfrak{s}}\Lambda(x)$ is given by the equation [2] :

$$^{P}D_{\mathfrak{s}}\Lambda(x) = K_1(\mathfrak{s}, \mathfrak{m})\Lambda(\mathfrak{m}) + K_0(\mathfrak{s}, \mathfrak{m})\Lambda'(\mathfrak{m}).$$

In this equation, K_1 and K_0 are the functions with respect to $\mathfrak{s} \in [0, 1]$ and $\mathfrak{m} \in \mathbb{R}$ subject to certain conditions and also, the function Λ is differentiable with respect to $\mathfrak{m} \in \mathbb{R}$. Research on the proportional derivative and the Caputo derivative has been increasingly significant in recent years [16, 18, 28–30].

Baleanu et al. provided the following definition in [4], wherein they creatively combined the ideas of proportional derivative and Caputo derivative to create a hybrid fractional operator that can be expressed as a linear combination of Riemann-Liouville fractional integral and Caputo fractional derivative.

Definition 1.3. Let $\Lambda : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I° and Λ, Λ' are locally $L_1(I)$. Then, the proportional Caputo-hybrid operator may be defined as follows:

$$^{C}D^{\mathfrak{s}}_{\psi+}\Lambda(\mathfrak{m}) = \frac{1}{\Gamma(1-\mathfrak{s})}\int_{0}^{\mathfrak{m}}\left[K_1(\mathfrak{s},\tau)\Lambda(\tau) + K_0(\mathfrak{s},\tau)\Lambda'(\tau)\right](\mathfrak{m}-\tau)^{-\mathfrak{s}}d\tau,$$

where $\mathfrak{s} \in [0, 1]$ and K_1 and K_0 are functions which satisfy the following conditions:

$$\lim_{\mathfrak{s}\to 0^+} K_0(\mathfrak{s},\tau) = 0; \quad \lim_{\mathfrak{s}\to 1} K_0(\mathfrak{s},\tau) = 1; \quad K_0(\mathfrak{s},\tau) \neq 0, \quad \mathfrak{s} \in (0, 1];$$
$$\lim_{\mathfrak{s}\to 0} K_1(\mathfrak{s},\tau) = 0; \quad \lim_{\mathfrak{s}\to 1^-} K_1(\mathfrak{s},\tau) = 1; \quad K_1(\mathfrak{s},\tau) \neq 0, \quad \mathfrak{s} \in [0, 1).$$

Alternatively, Sarıkaya [34] suggested a new notion by employing distinct K_1 and K_0 functions, guided by Definition 1.3, and subsequently demonstrated the Hermite-Hadamard inequality utilizing this definition:

Definition 1.4. Let $\Lambda : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I° and Λ, $\Lambda' \in L_1(I)$. The left-sided and right-sided proportional Caputo-hybrid operator of order $\mathfrak{s}$ are defined respectively as follows:

$$ {}^{PC}_{\psi^+}D^{\mathfrak{s}}_{\varphi}\ \Lambda(\varphi) = \frac{1}{\Gamma(1-\mathfrak{s})} \int\limits_{\psi}^{\varphi} \left[K_1(\mathfrak{s}, \varphi-\tau)\Lambda(\tau) + K_0(\mathfrak{s}, \varphi-\tau)\Lambda'(\tau)\right](\varphi-\tau)^{-\mathfrak{s}} d\tau $$

and

$$ {}^{PC}_{\varphi^-}D^{\mathfrak{s}}_{\psi}\ \Lambda(\psi) = \frac{1}{\Gamma(1-\mathfrak{s})} \int\limits_{\psi}^{\varphi} \left[K_1(\mathfrak{s}, \tau-\psi)\Lambda(\tau) + K_0(\mathfrak{s}, \tau-\psi)\Lambda'(\tau)\right](\tau-\psi)^{-\mathfrak{s}} d\tau, $$

where $\mathfrak{s} \in [0,1]$ and $K_0(\mathfrak{s},\tau) = (1-\mathfrak{s})^2\tau^{1-\mathfrak{s}}$ and $K_1(\mathfrak{s},\tau) = \mathfrak{s}^2\tau^{\mathfrak{s}}$.

Theorem 1.2. *Let $\Lambda : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I°, the interior of the interval I, where $\psi, \varphi \in I^\circ$ with $\psi < \varphi$ and Λ, Λ' be the convex functions on I. Then, the following inequalities hold:*

$$ \begin{aligned} &\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}\Lambda\left(\frac{\psi+\varphi}{2}\right) + \frac{1}{2}(1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}}\Lambda'\left(\frac{\psi+\varphi}{2}\right) \\ &\leq \frac{\Gamma(1-\mathfrak{s})}{2(\varphi-\psi)^{1-\mathfrak{s}}}\left[{}^{PC}_{\psi^+}D^{\mathfrak{s}}_{\varphi}\ \Lambda(\varphi) + {}^{PC}_{\varphi^-}D^{\mathfrak{s}}_{\psi}\ \Lambda(\psi)\right] \\ &\leq \mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}\left[\frac{\Lambda(\psi)+\Lambda(\varphi)}{2}\right] + (1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}}\left[\frac{\Lambda'(\psi)+\Lambda'(\varphi)}{4}\right]. \end{aligned} $$

Motivated by these studies, we initially present an alternative form of the Hermite-Hadamard inequality using the Caputo-hybrid operator introduced by Sarıkaya [34]. Additionally, we derive an identity for the trapezoid side of the Hermite-Hadamard inequality. Then, we obtain several significant inequalities by employing convexity, the Hölder inequality, and the power mean inequality. Finally, we offer several illustrative examples accompanied by graphical representations to showcase the derived inequalities.

2. Main results

In the following section, we introduce a novel Hermite-Hadamard inequality for the proportional Caputo-hybrid operator, which differs from the existing approaches.

Theorem 2.1. *Let $\Lambda : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I^o, the interior of the interval I, where $\psi, \varphi \in I^o$ satisfying $\psi < \varphi$ and Λ, Λ' be the convex functions on I. Then, the following inequalities are satisfied:*

$$ \mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\Lambda\left(\frac{\psi+\varphi}{2}\right) + (1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-2}\Lambda'\left(\frac{\psi+\varphi}{2}\right) \tag{2.1} $$

$$\leq \frac{\Gamma(1-\mathfrak{s})}{2^{\mathfrak{s}}(\varphi-\psi)^{-\mathfrak{s}+1}}\left[{}^{PC}_{\psi^+}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)+{}^{PC}_{\varphi^-}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)\right]$$
$$\leq \mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\left[\frac{\Lambda(\psi)+\Lambda(\varphi)}{2}\right]+(1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-2}\left[\frac{\Lambda'(\psi)+\Lambda'(\varphi)}{2}\right].$$

Proof. Due to the convexity property of Λ, Λ' on the interval $[\psi, \varphi]$, we get

$$\Lambda\left(\frac{\psi+\varphi}{2}\right)\leq\frac{1}{2}\left[\Lambda\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)+\Lambda\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)\right]$$

and

$$\Lambda'\left(\frac{\psi+\varphi}{2}\right)\leq\frac{1}{2}\left[\Lambda'\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)+\Lambda'\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)\right]$$

for $\mathfrak{m}\in[0,1]$. By multiplying the first expression by $\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}$ and the second expression by $(1-\mathfrak{s})^2(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-2\mathfrak{s}}$, we obtain

$$\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\Lambda\left(\frac{\psi+\varphi}{2}\right)$$
$$\leq\frac{1}{2}\left[\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\Lambda\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)+\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\Lambda\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)\right]$$

and

$$(1-\mathfrak{s})^2(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-2\mathfrak{s}}\Lambda'\left(\frac{\psi+\varphi}{2}\right)$$
$$\leq\frac{1}{2}\left[(1-\mathfrak{s})^2(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-2\mathfrak{s}}\Lambda'\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)\right.$$
$$\left.+(1-\mathfrak{s})^2(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-2\mathfrak{s}}\Lambda'\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)\right].$$

The result of combining these two statements side by side and adding them up is

$$\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\Lambda\left(\frac{\psi+\varphi}{2}\right)+(1-\mathfrak{s})^2(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-2\mathfrak{s}}\Lambda'\left(\frac{\psi+\varphi}{2}\right)$$
$$\leq\frac{1}{2}\left[\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\mathfrak{m}^{\mathfrak{s}}\Lambda\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)\right.$$
$$\left.+(1-\mathfrak{s})^2(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-\mathfrak{s}}\Lambda'\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)\right]\mathfrak{m}^{-\mathfrak{s}}$$
$$+\frac{1}{2}\left[\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\mathfrak{m}^{\mathfrak{s}}\Lambda\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)\right.$$
$$\left.+(1-\mathfrak{s})^2(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-\mathfrak{s}}\Lambda'\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)\right]\mathfrak{m}^{-\mathfrak{s}}.$$

By carrying out the integration of the inequality with respect to the parameter $\mathfrak{m}$ over the interval $[0,1]$, we can infer that

$$\begin{aligned}
&\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\Lambda\left(\frac{\psi+\varphi}{2}\right)+(1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-2}\Lambda'\left(\frac{\psi+\varphi}{2}\right)\\
\leq\ &\frac{1}{2}\int_0^1\left[\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\mathfrak{m}^{\mathfrak{s}}\Lambda\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)\right.\\
&\left.+(1-\mathfrak{s})^2(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-\mathfrak{s}}\Lambda'\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)\right]\mathfrak{m}^{-\mathfrak{s}}d\mathfrak{m}\\
&+\frac{1}{2}\int_0^1\left[\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\mathfrak{m}^{\mathfrak{s}}\Lambda\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)\right.\\
&\left.+(1-\mathfrak{s})^2(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-\mathfrak{s}}\Lambda'\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)\right]\mathfrak{m}^{-\mathfrak{s}}d\mathfrak{m}.
\end{aligned}$$

Through the use of a variable substitution, we derive the result that

$$\begin{aligned}
&\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\Lambda\left(\frac{\psi+\varphi}{2}\right)+(1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-2}\Lambda'\left(\frac{\psi+\varphi}{2}\right)\\
\leq\ &\frac{2^{-\mathfrak{s}}}{(\varphi-\psi)^{-\mathfrak{s}+1}}\\
&\times\int_{\psi}^{\frac{\psi+\varphi}{2}}\left[\mathfrak{s}^2\left(\frac{\psi+\varphi}{2}-\tau\right)^{\mathfrak{s}}\Lambda(\tau)+(1-\mathfrak{s})^2\left(\frac{\psi+\varphi}{2}-\tau\right)^{1-\mathfrak{s}}\Lambda'(\tau)\right]\left(\frac{\psi+\varphi}{2}-\tau\right)^{-\mathfrak{s}}d\tau\\
&+\frac{2^{-\mathfrak{s}}}{(\varphi-\psi)^{-\mathfrak{s}+1}}\\
&\times\int_{\frac{\psi+\varphi}{2}}^{\varphi}\left[\mathfrak{s}^2\left(\tau-\frac{\psi+\varphi}{2}\right)^{\mathfrak{s}}\Lambda(\tau)+(1-\mathfrak{s})^2\left(\tau-\frac{\psi+\varphi}{2}\right)^{1-\mathfrak{s}}\Lambda'(\tau)\right]\left(\tau-\frac{\psi+\varphi}{2}\right)^{-\mathfrak{s}}d\tau\\
=\ &\frac{2^{-\mathfrak{s}}}{(\varphi-\psi)^{-\mathfrak{s}+1}}\\
&\times\int_{\psi}^{\frac{\psi+\varphi}{2}}\left[K_1\left(\mathfrak{s},\frac{\psi+\varphi}{2}-\tau\right)\Lambda(\tau)+K_0\left(\mathfrak{s},\frac{\psi+\varphi}{2}-\tau\right)\Lambda'(\tau)\right]\left(\frac{\psi+\varphi}{2}-\tau\right)^{-\mathfrak{s}}d\tau\\
&+\frac{2^{-\mathfrak{s}}}{(\varphi-\psi)^{-\mathfrak{s}+1}}\\
&\times\int_{\frac{\psi+\varphi}{2}}^{\varphi}\left[K_1\left(\mathfrak{s},\tau-\frac{\psi+\varphi}{2}\right)\Lambda(\tau)+K_0\left(\mathfrak{s},\tau-\frac{\psi+\varphi}{2}\right)\Lambda'(\tau)\right]\left(\tau-\frac{\psi+\varphi}{2}\right)^{-\mathfrak{s}}d\tau\\
=\ &\frac{\Gamma(1-\mathfrak{s})}{2^{\mathfrak{s}}(\varphi-\psi)^{-\mathfrak{s}+1}}\left[{}^{PC}_{\psi^+}D^{\mathfrak{s}}_{(\frac{\psi+\varphi}{2})}\Lambda\left(\frac{\psi+\varphi}{2}\right)+{}^{PC}_{\varphi^-}D^{\mathfrak{s}}_{(\frac{\psi+\varphi}{2})}\Lambda\left(\frac{\psi+\varphi}{2}\right)\right].
\end{aligned}$$

Hence, the initial part of inequality (2.1) is shown. To establish the validity of the second side of (2.1), it can be inferred from the convexity of Λ and Λ'on the interval

$[\psi, \varphi]$ that

$$\Lambda\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)+\Lambda\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)\leq\Lambda(\psi)+\Lambda(\varphi)$$

and

$$\Lambda'\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)+\Lambda'\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)\leq\Lambda'(\psi)+\Lambda'(\varphi).$$

After multiplying the above two expressions by $\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}$ and $(1-\mathfrak{s})^2$ $(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-2\mathfrak{s}}$, respectively, we get

$$\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\Lambda\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)+\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\Lambda\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)$$

$$\leq\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\left[\Lambda(\psi)+\Lambda(\varphi)\right]$$

and

$$\begin{aligned}&(1-\mathfrak{s})^2(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-2\mathfrak{s}}\Lambda'\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)\\&+(1-\mathfrak{s})^2(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-2\mathfrak{s}}\Lambda'\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)\\\leq\;&(1-\mathfrak{s})^2(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-2\mathfrak{s}}\left[\Lambda'(\psi)+\Lambda'(\varphi)\right].\end{aligned}$$

So, adding up these two statements that are placed side by side gives us

$$\begin{aligned}&\frac{1}{2}\left[\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\mathfrak{m}^{\mathfrak{s}}\Lambda\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)\right.\\&\left.+(1-\mathfrak{s})^2(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-\mathfrak{s}}\Lambda'\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)\right]\mathfrak{m}^{-\mathfrak{s}}\\&+\frac{1}{2}\left[\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\mathfrak{m}^{\mathfrak{s}}\Lambda\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)\right.\\&\left.+(1-\mathfrak{s})^2(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-\mathfrak{s}}\Lambda'\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)\right]\mathfrak{m}^{-\mathfrak{s}}\\\leq\;&\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\left[\frac{\Lambda(\psi)+\Lambda(\varphi)}{2}\right]+(1-\mathfrak{s})^2(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-2\mathfrak{s}}\left[\frac{\Lambda'(\psi)+\Lambda'(\varphi)}{2}\right].\end{aligned}$$

By integrating both sides of the inequality using the parameter $\mathfrak{m}$ over $[0,1]$, we can infer that

$$\begin{aligned}&\frac{1}{2}\int_0^1\left[\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\mathfrak{m}^{\mathfrak{s}}\Lambda\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)\right.\\&\left.+(1-\mathfrak{s})^2(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-\mathfrak{s}}\Lambda'\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)\right]\mathfrak{m}^{-\mathfrak{s}}d\mathfrak{m}\end{aligned}$$

$$+\frac{1}{2}\int_0^1 \left[\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\mathfrak{m}^{\mathfrak{s}}\Lambda\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)\right.$$

$$\left.+(1-\mathfrak{s})^2(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-1}\mathfrak{m}^{1-\mathfrak{s}}\Lambda'\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)\right]\mathfrak{m}^{-\mathfrak{s}}d\mathfrak{m}$$

$$\leq \mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\left[\frac{\Lambda(\psi)+\Lambda(\varphi)}{2}\right]+(1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-2}\left[\frac{\Lambda'(\psi)+\Lambda'(\varphi)}{2}\right].$$

Thus, by substituting variables, we can deduce the conclusion

$$\frac{\Gamma(1-\mathfrak{s})}{2^{\mathfrak{s}}(\varphi-\psi)^{-\mathfrak{s}+1}}\left[{}^{PC}_{\psi^+}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)+{}^{PC}_{\varphi^-}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)\right]$$

$$\leq \mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\left[\frac{\Lambda(\psi)+\Lambda(\varphi)}{2}\right]+(1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-2}\left[\frac{\Lambda'(\psi)+\Lambda'(\varphi)}{2}\right].$$

As a result, we achieve the required second side of inequality (2.1). □

We now offer an example that clearly demonstrates the superiority of our theorem.

Example 2.1. Let us take a function $\Lambda:[0,2]\to\mathbb{R}$ defined as $\Lambda(\pi)=\pi^3+\pi^2$. Then, we establish that the right-hand side and left-hand side of (2.1) are

$$6\mathfrak{s}^2+4(1-\mathfrak{s}) \quad \text{and} \quad 2\mathfrak{s}^2+\frac{5}{2}(1-\mathfrak{s}).$$

Moreover, we have

$$\frac{\Gamma(1-\mathfrak{s})}{2^{\mathfrak{s}}(\varphi-\psi)^{-\mathfrak{s}+1}}\left[{}^{PC}_{\psi^+}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)+{}^{PC}_{\varphi^-}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)\right]$$

$$=\frac{10}{3}\mathfrak{s}^2+\frac{(1-\mathfrak{s})^2}{2}\left(\frac{10}{2-2\mathfrak{s}}+\frac{6}{4-2\mathfrak{s}}\right).$$

Thus, according to inequality (2.1), we arrive at the inequality

$$2\mathfrak{s}^2+\frac{5}{2}(1-\mathfrak{s})\leq\frac{10}{3}\mathfrak{s}^2+\frac{(1-\mathfrak{s})^2}{2}\left(\frac{10}{2-2\mathfrak{s}}+\frac{6}{4-2\mathfrak{s}}\right)\leq 6\mathfrak{s}^2+4(1-\mathfrak{s}). \tag{2.2}$$

Figure 1 visually demonstrates the validity of inequality (2.2).

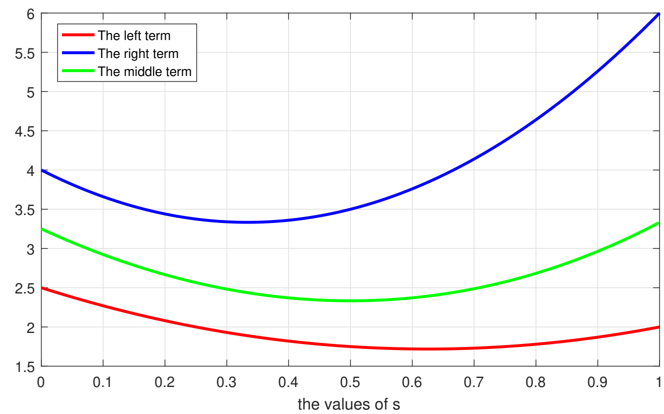

Figure 1. The graph of three parts of the inequality (2.2) in Example 2.1, which is computed and drawn in MATLAB program, depending on $\mathfrak{s} \in (0,1)$

The following lemma is crucial for the proof of our other theorems. Therefore, let us begin by presenting the proof of this lemma.

Lemma 2.1. *Let $\Lambda : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I^o, the interior of the interval I, where $\psi, \varphi \in I^o$ satisfying $\psi < \varphi$ and let $\Lambda, \Lambda', \Lambda'' \in L_1[\psi, \varphi]$. Then, the following identity is satisfied:*

$$\begin{aligned}
&\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}+1}2^{-\mathfrak{s}-1}\int_0^1 (1-2\mathfrak{m})\Lambda'\left(\mathfrak{m}\psi+(1-\mathfrak{m})\varphi\right)d\mathfrak{m} \\
&+(1-\mathfrak{s})(\varphi-\psi)^{2-\mathfrak{s}}2^{\mathfrak{s}-4}\int_0^1 \mathfrak{m}^{2-2\mathfrak{s}}\left[\Lambda''\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)\right. \\
&\left.-\Lambda''\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)\right]d\mathfrak{m} \\
&=\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\left(\frac{\Lambda(\psi)+\Lambda(\varphi)}{2}\right)+(1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-2}\left(\frac{\Lambda'(\psi)+\Lambda'(\varphi)}{2}\right) \\
&-\frac{\Gamma(1-\mathfrak{s})}{2^{\mathfrak{s}}(\varphi-\psi)^{-\mathfrak{s}+1}}\left[{}^{PC}_{\psi^+}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)+{}^{PC}_{\varphi^-}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)\right].
\end{aligned} \tag{2.3}$$

Proof. By integration by parts, we have

$$\int_0^1 \mathfrak{m}\Lambda'\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)d\mathfrak{m}=\frac{2}{\varphi-\psi}\Lambda(\varphi)-\frac{2}{\varphi-\psi}\int_0^1\Lambda\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)d\mathfrak{m}$$

and

$$\begin{aligned}
\int_0^1 \mathfrak{m}^{2-2\mathfrak{s}}\Lambda''\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)d\mathfrak{m}&=\frac{2\Lambda'(\varphi)}{\varphi-\psi} \\
&-\frac{4(1-\mathfrak{s})}{\varphi-\psi}\int_0^1 \mathfrak{m}^{1-2\mathfrak{s}}\Lambda'\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)d\mathfrak{m}.
\end{aligned}$$

By employing a change of variable, multiplying the results by $\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}+1}2^{-\mathfrak{s}-1}$ and $(1-\mathfrak{s})(\varphi-\psi)^{2-\mathfrak{s}}2^{\mathfrak{s}-3}$, and combining them side by side, we obtain the following outcome:

$$\begin{aligned}
&\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}+1}2^{-\mathfrak{s}-1}\int_0^1 \mathfrak{m}\Lambda'\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)d\mathfrak{m} \\
&+(1-\mathfrak{s})(\varphi-\psi)^{2-\mathfrak{s}}2^{\mathfrak{s}-3}\int_0^1 \mathfrak{m}^{2-2\mathfrak{s}}\Lambda''\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)d\mathfrak{m} \\
=\;&\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\Lambda(\varphi)+(1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-2}\Lambda'(\varphi) \\
&-\frac{2^{1-\mathfrak{s}}}{(\varphi-\psi)^{1-\mathfrak{s}}} \\
&\times\int_{\frac{\psi+\varphi}{2}}^{\varphi}\left[\mathfrak{s}^2\left(\tau-\frac{\psi+\varphi}{2}\right)^{\mathfrak{s}}\Lambda(\tau)+(1-\mathfrak{s})^2\left(\tau-\frac{\psi+\varphi}{2}\right)^{1-\mathfrak{s}}\Lambda'(\tau)\right]\left(\tau-\frac{\psi+\varphi}{2}\right)^{-\mathfrak{s}}d\tau.
\end{aligned} \tag{2.4}$$

Another result obtained through analogous steps is as follows:

$$\begin{aligned}
&\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}+1}2^{-\mathfrak{s}-1}\int_0^1 \mathfrak{m}\Lambda'\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)d\mathfrak{m} \\
&+(1-\mathfrak{s})(\varphi-\psi)^{2-\mathfrak{s}}2^{\mathfrak{s}-3}\int_0^1 \mathfrak{m}^{2-2\mathfrak{s}}\Lambda''\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)d\mathfrak{m} \\
=\;&-\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\Lambda(\psi)-(1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-2}\Lambda'(\psi) \\
&+\frac{2^{1-\mathfrak{s}}}{(\varphi-\psi)^{1-\mathfrak{s}}} \\
&\times\int_{\psi}^{\frac{\psi+\varphi}{2}}\left[\mathfrak{s}^2\left(\frac{\psi+\varphi}{2}-\tau\right)^{\mathfrak{s}}\Lambda(\tau)+(1-\mathfrak{s})^2\left(\frac{\psi+\varphi}{2}-\tau\right)^{1-\mathfrak{s}}\Lambda'(\tau)\right]\left(\frac{\psi+\varphi}{2}-\tau\right)^{-\mathfrak{s}}d\tau.
\end{aligned} \tag{2.5}$$

Subtracting (2.5) from (2.4), we have

$$\begin{aligned}
&\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}+1}2^{-\mathfrak{s}-1}\int_0^1 \mathfrak{m}\left[\Lambda'\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)-\Lambda'\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)\right]d\mathfrak{m} \\
&+(1-\mathfrak{s})(\varphi-\psi)^{2-\mathfrak{s}}2^{\mathfrak{s}-3} \\
&\times\int_0^1 \mathfrak{m}^{2-2\mathfrak{s}}\left[\Lambda''\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)-\Lambda''\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)\right]d\mathfrak{m} \\
=\;&\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\left[\Lambda(\psi)+\Lambda(\varphi)\right]+(1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-2}\left[\Lambda'(\psi)+\Lambda'(\varphi)\right] \\
&-\frac{\Gamma(1-\mathfrak{s})}{2^{\mathfrak{s}-1}(\varphi-\psi)^{-\mathfrak{s}+1}}\left[{}_{\psi^+}^{PC}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)+{}_{\varphi^-}^{PC}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)\right].
\end{aligned}$$

Thus, by multiplying both sides by $\frac{1}{2}$ and by using the equality

$$\int_0^1 \mathfrak{m}\left[\Lambda'\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)-\Lambda'\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)\right]d\mathfrak{m}$$
$$=2\int_0^1 (1-2\mathfrak{m})\Lambda'\left(\mathfrak{m}\psi+(1-\mathfrak{m})\varphi\right)d\mathfrak{m},$$

we conclude the proof. □

Remark 2.1. Letting the limit as $\mathfrak{s}\to 1$ in Lemma 2.1, it follows that

$$\frac{\Lambda(\psi)+\Lambda(\varphi)}{2}-\frac{1}{\varphi-\psi}\int_\psi^\varphi \Lambda(\pi)d\pi=\frac{(\varphi-\psi)}{2}\int_0^1(1-2\mathfrak{m})\Lambda'(\mathfrak{m}\psi+(1-\mathfrak{m})\varphi)d\mathfrak{m},$$

which was proved by Dragomir and Agarwal [13].

Corollary 2.1. *In the limiting case $\mathfrak{s}=0$ in Lemma 2.1, we have*

$$\frac{(\varphi-\psi)^2}{16}\left(\int_0^1 \mathfrak{m}^2\left[\Lambda''\left(\frac{1-\mathfrak{m}}{2}\psi+\frac{1+\mathfrak{m}}{2}\varphi\right)-\Lambda''\left(\frac{1+\mathfrak{m}}{2}\psi+\frac{1-\mathfrak{m}}{2}\varphi\right)\right]d\mathfrak{m}\right)$$
$$=\frac{(\varphi-\psi)}{4}\left(\frac{\Lambda'(\psi)+\Lambda'(\varphi)}{2}\right)-\frac{\Lambda(\varphi)-\Lambda(\psi)}{2}-\frac{1}{\varphi-\psi}\left(\int_\psi^{\frac{\psi+\varphi}{2}}\Lambda(\pi)d\pi-\int_{\frac{\psi+\varphi}{2}}^{\varphi}\Lambda(\pi)d\pi\right).$$

In addition, when $\mathfrak{s}=\frac{1}{2}$ is chosen, the equality (2.3) takes the form of the following equality:

$$\frac{2}{\varphi-\psi}\left\{\frac{\Lambda(\psi)+\Lambda(\varphi)}{2}+\frac{\Lambda'(\psi)+\Lambda'(\varphi)}{2}-\frac{1}{\varphi-\psi}\left[\int_\psi^\varphi\Lambda(\pi)d\pi+\Lambda(\varphi)-\Lambda(\psi)\right]\right\}$$
$$=\int_0^1(1-2\mathfrak{m})\left[\Lambda'(\mathfrak{m}\psi+(1-\mathfrak{m})\varphi)+\Lambda''(\mathfrak{m}\psi+(1-\mathfrak{m})\varphi)\right]d\mathfrak{m}.$$

Theorem 2.2. *Let $\Lambda: I\subset\mathbb{R}^+\to\mathbb{R}$ be a twice differentiable function on I^o, the interior of the interval I, where $\psi,\varphi\in I^o$ satisfying $\psi<\varphi$ and let $\Lambda,\Lambda',\Lambda''\in L_1[\psi,\varphi]$. If $|\Lambda'|^q$ and $|\Lambda''|^q$ are convex on $[\psi,\varphi]$ for $q\geq 1$, then the following inequality holds:*

$$\left|\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\left(\frac{\Lambda(\psi)+\Lambda(\varphi)}{2}\right)+(1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-2}\left(\frac{\Lambda'(\psi)+\Lambda'(\varphi)}{2}\right)\right.$$
$$\left.-\frac{\Gamma(1-\mathfrak{s})}{2^{\mathfrak{s}}(\varphi-\psi)^{-\mathfrak{s}+1}}\left[{}^{PC}_{\psi^+}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)+{}^{PC}_{\varphi^-}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)\right]\right| \tag{2.6}$$

$$\leq \frac{\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}+1}2^{-\mathfrak{s}}}{4}\left(\frac{|\Lambda'(\psi)|^q+|\Lambda'(\varphi)|^q}{2}\right)^{\frac{1}{q}}$$
$$+\frac{(1-\mathfrak{s})(\varphi-\psi)^{2-\mathfrak{s}}2^{\mathfrak{s}}}{16}\left\{\frac{1}{(3-2\mathfrak{s})^{\frac{q-1}{q}}}\left(\left[\frac{|\Lambda''(\psi)|^q}{2}\left(\frac{1}{(3-2\mathfrak{s})(4-2\mathfrak{s})}\right)\right.\right.\right.$$
$$\left.+\frac{|\Lambda''(\varphi)|^q}{2}\left(\frac{7-4\mathfrak{s}}{(3-2\mathfrak{s})(4-2\mathfrak{s})}\right)\right]^{\frac{1}{q}}+\left[\frac{|\Lambda''(\psi)|^q}{2}\left(\frac{7-4\mathfrak{s}}{(3-2\mathfrak{s})(4-2\mathfrak{s})}\right)\right.$$
$$\left.\left.\left.+\frac{|\Lambda''(\varphi)|^q}{2}\left(\frac{1}{(3-2\mathfrak{s})(4-2\mathfrak{s})}\right)\right]^{\frac{1}{q}}\right)\right\}.$$

Proof. At first, let us consider the case where $q=1$. By utilizing the convexity of $|\Lambda'|$ and $|\Lambda''|$, we obtain from Lemma 2.1 that

$$\left|\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\left(\frac{\Lambda(\psi)+\Lambda(\varphi)}{2}\right)+(1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-2}\left(\frac{\Lambda'(\psi)+\Lambda'(\varphi)}{2}\right)\right. \tag{2.7}$$
$$\left.-\frac{\Gamma(1-\mathfrak{s})}{2^{\mathfrak{s}}(\varphi-\psi)^{-\mathfrak{s}+1}}\left[{}_{\psi^+}^{PC}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)+{}_{\varphi^-}^{PC}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)\right]\right|$$
$$\leq \frac{\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}+1}2^{-\mathfrak{s}}}{2}\int_0^1|1-2\mathfrak{m}|\left(\mathfrak{m}|\Lambda'(\psi)|+(1-\mathfrak{m})|\Lambda'(\varphi)|\right)d\mathfrak{m}$$
$$+\frac{(1-\mathfrak{s})(\varphi-\psi)^{2-\mathfrak{s}}2^{\mathfrak{s}}}{16}\int_0^1\mathfrak{m}^{2-2\mathfrak{s}}\left(\frac{1-\mathfrak{m}}{2}\left|\Lambda''(\psi)\right|+\frac{1+\mathfrak{m}}{2}\left|\Lambda''(\varphi)\right|+\frac{1+\mathfrak{m}}{2}\left|\Lambda''(\psi)\right|\right.$$
$$\left.+\frac{1-\mathfrak{m}}{2}\left|\Lambda''(\varphi)\right|\right)d\mathfrak{m}.$$

Therefore, from the fact that

$$\int_0^{\frac{1}{2}}(1-2\mathfrak{m})\mathfrak{m}d\mathfrak{m}=\int_{\frac{1}{2}}^1(2\mathfrak{m}-1)(1-\mathfrak{m})d\mathfrak{m}=\frac{1}{24},$$
$$\int_0^{\frac{1}{2}}(1-2\mathfrak{m})(1-\mathfrak{m})d\mathfrak{m}=\int_{\frac{1}{2}}^1(2\mathfrak{m}-1)\mathfrak{m}d\mathfrak{m}=\frac{5}{24}$$

and

$$\int_0^1\mathfrak{m}^{2-2\mathfrak{s}}(1-\mathfrak{m})\,d\mathfrak{m}=\frac{1}{(3-2\mathfrak{s})(4-2\mathfrak{s})},$$
$$\int_0^1\mathfrak{m}^{2-2\mathfrak{s}}(1+\mathfrak{m})d\mathfrak{m}=\frac{7-4\mathfrak{s}}{(3-2\mathfrak{s})(4-2\mathfrak{s})},$$

it follows that the expression on the right-hand side of inequality (2.7) is

$$\frac{\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}+1}2^{-\mathfrak{s}}}{4}\left(\frac{|\Lambda'(\psi)|+|\Lambda'(\varphi)|}{2}\right)+\frac{(1-\mathfrak{s})(\varphi-\psi)^{2-\mathfrak{s}}2^{\mathfrak{s}}}{8(3-2\mathfrak{s})}\left(\frac{|\Lambda''(\psi)|+|\Lambda''(\varphi)|}{2}\right).$$

Additionally, in the case of $q > 1$, by employing Lemma 2.1, the power mean inequality, and taking into account the convexity of $|\Lambda'|^q$ and $|\Lambda''|^q$, we can deduce:

$$\begin{aligned}
&\Bigg|\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\left(\frac{\Lambda(\psi)+\Lambda(\varphi)}{2}\right)+(1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-2}\left(\frac{\Lambda'(\psi)+\Lambda'(\varphi)}{2}\right)\\
&-\frac{\Gamma(1-\mathfrak{s})}{2^{\mathfrak{s}}(\varphi-\psi)^{-\mathfrak{s}+1}}\left[{}^{PC}_{\psi^+}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)+{}^{PC}_{\varphi^-}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)\right]\Bigg|\\
\leq\ &\frac{\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}+1}2^{-\mathfrak{s}}}{2}\\
&\times\left\{\left(\int_0^1|1-2\mathfrak{m}|\,d\mathfrak{m}\right)^{\frac{1}{p}}\left(\int_0^1|1-2\mathfrak{m}|\left[\mathfrak{m}|\Lambda'(\psi)|^q+(1-\mathfrak{m})|\Lambda'(\varphi)|^q\right]d\mathfrak{m}\right)^{\frac{1}{q}}\right\}\\
&+\frac{(1-\mathfrak{s})(\varphi-\psi)^{2-\mathfrak{s}}2^{\mathfrak{s}}}{16}\\
&\times\left\{\left(\int_0^1\mathfrak{m}^{2-2\mathfrak{s}}d\mathfrak{m}\right)^{\frac{1}{p}}\left(\int_0^1\mathfrak{m}^{2-2\mathfrak{s}}\left[\frac{1-\mathfrak{m}}{2}|\Lambda''(\psi)|^q+\frac{1+\mathfrak{m}}{2}|\Lambda''(\varphi)|^q\right]d\mathfrak{m}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\mathfrak{m}^{2-2\mathfrak{s}}d\mathfrak{m}\right)^{\frac{1}{p}}\left(\int_0^1\mathfrak{m}^{2-2\mathfrak{s}}\left[\frac{1+\mathfrak{m}}{2}|\Lambda''(\psi)|^q+\frac{1-\mathfrak{m}}{2}|\Lambda''(\varphi)|^q\right]d\mathfrak{m}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Thus, due to

$$\int_0^1|1-2\mathfrak{m}|\,d\mathfrak{m}=\int_0^{\frac{1}{2}}(1-2\mathfrak{m})\,d\mathfrak{m}+\int_{\frac{1}{2}}^1(2\mathfrak{m}-1)\,d\mathfrak{m}=\frac{1}{2},$$

$$\int_0^1\mathfrak{m}^{2-2\mathfrak{s}}d\mathfrak{m}=\frac{1}{3-2\mathfrak{s}},$$

and the desired inequality (2.6) holds. □

Remark 2.2. Using $\mathfrak{s}\to 1$ as the limit and $q=1$ in Theorem 2.2, it can be deduced that

$$\left|\frac{\Lambda(\psi)+\Lambda(\varphi)}{2}-\frac{1}{\varphi-\psi}\int_\psi^\varphi\Lambda(\pi)d\pi\right|\leq\frac{(\varphi-\psi)}{4}\left(\frac{|\Lambda'(\psi)|+|\Lambda'(\varphi)|}{2}\right),$$

which was proved by Dragomir and Agarwal [13].

Corollary 2.2. *For $q\geq 1$ in Theorem 2.2 and as $\mathfrak{s}\to 0$, we obtain*

$$\left|\frac{\varphi-\psi}{4}\left(\frac{\Lambda'(\psi)+\Lambda'(\varphi)}{2}\right)-\Lambda(\varphi)+\Lambda(\psi)-\frac{1}{\varphi-\psi}\left(\int_\psi^{\frac{\psi+\varphi}{2}}\Lambda(\pi)d\pi-\int_{\frac{\psi+\varphi}{2}}^\varphi\Lambda(\pi)d\pi\right)\right|$$

$$\leq \frac{(\varphi-\psi)^2}{16\left(3^{\frac{q-1}{q}}\right)}\left[\left(\frac{1}{24}\left|\Lambda''(\psi)\right|^q+\frac{7}{24}\left|\Lambda''(\varphi)\right|^q\right)^{1/q}+\left(\frac{7}{24}\left|\Lambda''(\psi)\right|^q+\frac{1}{24}\left|\Lambda''(\varphi)\right|^q\right)^{1/q}\right].$$

Furthermore, when $\mathfrak{s}$ converges to 1 and with $q \geq 1$, the inequality described in Theorem 2.2 is given by

$$\left|\frac{\Lambda(\psi)+\Lambda(\varphi)}{2}-\frac{1}{\varphi-\psi}\int_{\psi}^{\varphi}\Lambda(\pi)d\pi\right| \leq \frac{(\varphi-\psi)}{4}\left(\frac{\left|\Lambda'(\psi)\right|^q+\left|\Lambda'(\varphi)\right|^q}{2}\right)^{1/q}.$$

In order to confirm the applicability of our theorem, we offer an illustrative example.

Example 2.2. Considering the function $\Lambda(\pi) = \pi^3$ defined on the interval $[0, 2]$, we can compute the right-hand side of inequality (2.6) as follows:

$$\frac{6\mathfrak{s}^2}{2^{\frac{1}{q}}}+\frac{(1-\mathfrak{s})}{4}\left(\frac{1}{3-2\mathfrak{s}}\right)^{\frac{q-1}{q}}\frac{12}{2^{\frac{1}{q}}}\left[\left(\frac{7-4\mathfrak{s}}{(3-2\mathfrak{s})(4-2\mathfrak{s})}\right)^{\frac{1}{q}}+\left(\frac{1}{(3-2\mathfrak{s})(4-2\mathfrak{s})}\right)^{\frac{1}{q}}\right].$$

On the other hand, we see that

$$\begin{aligned}&\left|\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\left(\frac{\Lambda(\psi)+\Lambda(\varphi)}{2}\right)+(1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-2}\left(\frac{\Lambda'(\psi)+\Lambda'(\varphi)}{2}\right)\right.\\&\left.-\frac{\Gamma(1-\mathfrak{s})}{2^{\mathfrak{s}}(\varphi-\psi)^{-\mathfrak{s}+1}}\left[{}^{PC}_{\psi^+}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)+{}^{PC}_{\varphi^-}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)\right]\right|\\&=2\mathfrak{s}^2+3(1-\mathfrak{s})-\frac{3}{2}(1-\mathfrak{s})^2\frac{3-2\mathfrak{s}}{(1-\mathfrak{s})(2-\mathfrak{s})}.\end{aligned}$$

For all $\mathfrak{s} \in (0, 1)$ and $q \geq 1$, it is evident from Figure 2 that the left-hand side of the inequality (2.6) is always located below the right-hand side of this inequality.

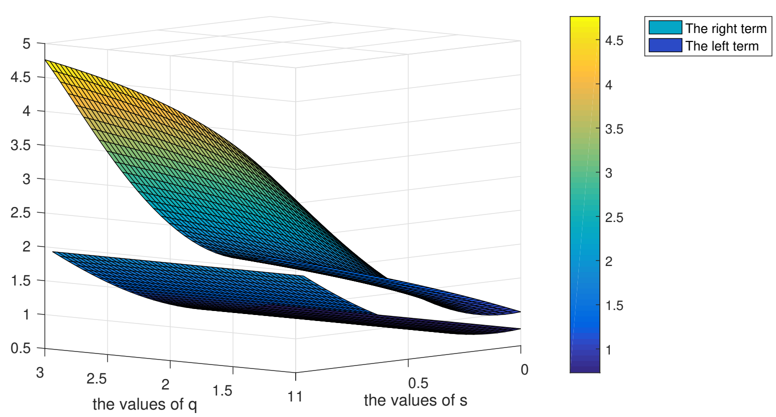

Figure 2. The graph of both sides of the inequality (2.6), which is computed and drawn in MATLAB program, depending on $\mathfrak{s} \in (0, 1)$ and $q \in [1, 3]$.

Theorem 2.3. *Let $\Lambda : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I^o, the interior of the interval I, where $\psi, \varphi \in I^o$ satisfying $\psi < \varphi$ and let $\Lambda, \Lambda', \Lambda'' \in L[\psi, \varphi]$. If $|\Lambda'|^q$ and $|\Lambda''|^q$ are convex on $[\psi, \varphi]$ for $q > 1$, then the following inequality holds:*

$$\begin{aligned}
&\Bigg| \mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}} 2^{-\mathfrak{s}} \left(\frac{\Lambda(\psi)+\Lambda(\varphi)}{2}\right) + (1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}} 2^{\mathfrak{s}-2} \left(\frac{\Lambda'(\psi)+\Lambda'(\varphi)}{2}\right) \\
&\quad - \frac{\Gamma(1-\mathfrak{s})}{2^{\mathfrak{s}}(\varphi-\psi)^{-\mathfrak{s}+1}} \left[{}^{PC}_{\psi^+}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)} \Lambda\left(\frac{\psi+\varphi}{2}\right) + {}^{PC}_{\varphi^-}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)} \Lambda\left(\frac{\psi+\varphi}{2}\right) \right] \Bigg| \\
&\leq \mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}+1} 2^{-\mathfrak{s}-1} \left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{|\Lambda'(\psi)|^q + |\Lambda'(\varphi)|^q}{2}\right)^{\frac{1}{q}} \right] \\
&\quad + (1-\mathfrak{s})(\varphi-\psi)^{2-\mathfrak{s}} 2^{\mathfrak{s}-4} \left(\frac{1}{2p-2\mathfrak{s}p+1}\right)^{\frac{1}{p}} \left[\left(\frac{|\Lambda''(\psi)|^q + 3|\Lambda''(\varphi)|^q}{4}\right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\frac{3|\Lambda''(\psi)|^q + |\Lambda''(\varphi)|^q}{4}\right)^{\frac{1}{q}} \right],
\end{aligned} \tag{2.8}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By applying the widely used Hölder's inequality and taking into account the convexity of $|\Lambda'|^q$, $|\Lambda''|^q$, based on Lemma 2.1, we obtain

$$\begin{aligned}
&\Bigg| \mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}} 2^{-\mathfrak{s}} \left(\frac{\Lambda(\psi)+\Lambda(\varphi)}{2}\right) + (1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}} 2^{\mathfrak{s}-2} \left(\frac{\Lambda'(\psi)+\Lambda'(\varphi)}{2}\right) \\
&\quad - \frac{\Gamma(1-\mathfrak{s})}{2^{\mathfrak{s}}(\varphi-\psi)^{-\mathfrak{s}+1}} \left[{}^{PC}_{\psi^+}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)} \Lambda\left(\frac{\psi+\varphi}{2}\right) + {}^{PC}_{\varphi^-}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)} \Lambda\left(\frac{\psi+\varphi}{2}\right) \right] \Bigg| \\
&\leq \mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}+1} 2^{-\mathfrak{s}-1} \\
&\quad \times \left[\left(\int_0^1 |1-2\mathfrak{m}|^p \, d\mathfrak{m} \right)^{\frac{1}{p}} \left(\int_0^1 \left[\mathfrak{m} |\Lambda'(\psi)|^q + (1-\mathfrak{m}) |\Lambda'(\varphi)|^q \right] d\mathfrak{m} \right)^{\frac{1}{q}} \right] \\
&\quad + (1-\mathfrak{s})(\varphi-\psi)^{2-\mathfrak{s}} 2^{\mathfrak{s}-4} \\
&\quad \times \left[\left(\int_0^1 \mathfrak{m}^{2p-2\mathfrak{s}p} d\mathfrak{m} \right)^{\frac{1}{p}} \left(\int_0^1 \left[\frac{1-\mathfrak{m}}{2} |\Lambda''(\psi)|^q + \frac{1+\mathfrak{m}}{2} |\Lambda''(\varphi)|^q \right] d\mathfrak{m} \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_0^1 \mathfrak{m}^{2p-2\mathfrak{s}p} d\mathfrak{m} \right)^{\frac{1}{p}} \left(\int_0^1 \left[\frac{1+\mathfrak{m}}{2} |\Lambda''(\psi)|^q + \frac{1-\mathfrak{m}}{2} |\Lambda''(\varphi)|^q \right] d\mathfrak{m} \right)^{\frac{1}{q}} \right].
\end{aligned} \tag{2.9}$$

Evaluating the integrals in the inequality mentioned above, we get

$$\int_0^1 |1-2\mathfrak{m}|^p \, d\mathfrak{m} = \int_0^{1/2} (1-2\mathfrak{m})^p d\mathfrak{m} + \int_{1/2}^1 (2\mathfrak{m}-1)^p d\mathfrak{m} = \frac{1}{p+1},$$

$$\int_0^1 \left[\mathfrak{m}\,|\Lambda'(\psi)|^q + (1-\mathfrak{m})\,|\Lambda'(\varphi)|^q\right] d\mathfrak{m} = \frac{|\Lambda'(\psi)|^q + |\Lambda'(\varphi)|^q}{2},$$

$$\int_0^1 \left[\frac{1-\mathfrak{m}}{2}\,|\Lambda''(\psi)|^q + \frac{1+\mathfrak{m}}{2}\,|\Lambda''(\varphi)|^q\right] d\mathfrak{m} = \frac{|\Lambda''(\psi)|^q + 3|\Lambda''(\varphi)|^q}{4},$$

$$\int_0^1 \left[\frac{1+\mathfrak{m}}{2}\,|\Lambda''(\psi)|^q + \frac{1-\mathfrak{m}}{2}\,|\Lambda''(\varphi)|^q\right] d\mathfrak{m} = \frac{3|\Lambda''(\psi)|^q + |\Lambda''(\varphi)|^q}{4}.$$

Thus, the desired result can be achieved by replacing the calculated integral results into the inequality (2.9). □

Remark 2.3. In Theorem 2.3, in the specific situation where $\mathfrak{s} \to 1$, we have

$$\left|\frac{\Lambda(\psi)+\Lambda(\varphi)}{2} - \frac{1}{\varphi-\psi}\int_\psi^\varphi \Lambda(\pi)d\pi\right| \leq \frac{\varphi-\psi}{2\,(p+1)^{1/p}}\left(\frac{|\Lambda'(\psi)|^q + |\Lambda'(\varphi)|^q}{2}\right)^{1/q},$$

which was proved by Dragomir and Agarwal in [13].

Corollary 2.3. *As $\mathfrak{s}$ approaches 0 in Theorem 2.3, we obtain the following particular case:*

$$\left|\frac{\varphi-\psi}{4}\left(\frac{\Lambda'(\psi)+\Lambda'(\varphi)}{2}\right)+\Lambda(\psi)-\Lambda(\varphi)-\frac{1}{\varphi-\psi}\left(\int_\psi^{\frac{\psi+\varphi}{2}} \Lambda(\pi)d\pi - \int_{\frac{\psi+\varphi}{2}}^{\varphi} \Lambda(\pi)d\pi\right)\right|$$
$$\leq \frac{(\varphi-\psi)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\Lambda''(\psi)|^q+3\,|\Lambda''(\varphi)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3\,|\Lambda''(\psi)|^q+|\Lambda''(\varphi)|^q}{4}\right)^{\frac{1}{q}}\right].$$

Furthermore, if we choose $\mathfrak{s} = \frac{1}{2}$, then we get

$$\left|\frac{\Lambda(\psi)+\Lambda(\varphi)}{2} + \frac{\Lambda'(\psi)+\Lambda'(\varphi)}{2} - \frac{1}{\varphi-\psi}\left[\int_\psi^\varphi \Lambda(\pi)d\pi + \Lambda(\varphi)-\Lambda(\psi)\right]\right|$$
$$\leq \frac{(\varphi-\psi)}{2(p+1)^{\frac{1}{p}}}\left(\frac{|\Lambda'(\psi)|^q+|\Lambda'(\varphi)|^q}{2}\right)^{\frac{1}{q}}$$
$$+\frac{\varphi-\psi}{4(p+1)^{\frac{1}{p}}}\left[\left(\frac{|\Lambda''(\psi)|^q+3|\Lambda''(\varphi)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|\Lambda''(\psi)|^q+|\Lambda''(\varphi)|^q}{4}\right)^{\frac{1}{q}}\right].$$

For the purpose of demonstrating the inequality established in Theorem 2.3, we offer an example that confirms its validity.

Example 2.3. In light of the function Λ given in Example 2.2, we can assess the expression on the right-hand side of the inequality (2.8) in the following manner:

$$\frac{12\mathfrak{s}^2}{2^{\frac{p-1}{p}}(p+1)^{\frac{1}{p}}}+\frac{12(1-\mathfrak{s})}{4^{\frac{2p-1}{p}}}\left(\frac{1}{(2-2\mathfrak{s})p+1}\right)^{\frac{1}{p}}\left(1+3^{\frac{p-1}{p}}\right).$$

Moreover, we know that

$$\left|\mathfrak{s}^2(\varphi-\psi)^{\mathfrak{s}}2^{-\mathfrak{s}}\left(\frac{\Lambda(\psi)+\Lambda(\varphi)}{2}\right)+(1-\mathfrak{s})(\varphi-\psi)^{1-\mathfrak{s}}2^{\mathfrak{s}-2}\left(\frac{\Lambda'(\psi)+\Lambda'(\varphi)}{2}\right)\right.$$
$$\left.-\frac{\Gamma(1-\mathfrak{s})}{2^{\mathfrak{s}}(\varphi-\psi)^{-\mathfrak{s}+1}}\left[{}^{PC}_{\psi^+}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)+{}^{PC}_{\varphi^-}D^{\mathfrak{s}}_{\left(\frac{\psi+\varphi}{2}\right)}\Lambda\left(\frac{\psi+\varphi}{2}\right)\right]\right|$$
$$=2\mathfrak{s}^2+3(1-\mathfrak{s})-\frac{3}{2}(1-\mathfrak{s})^2\frac{3-2\mathfrak{s}}{(1-\mathfrak{s})(2-\mathfrak{s})}.$$

Thus, Figure 3 clearly indicates that the left-hand side of inequality (2.8) is consistently less than the right-hand side for all values of $\mathfrak{s}\in(0,1)$ and $p>1$.

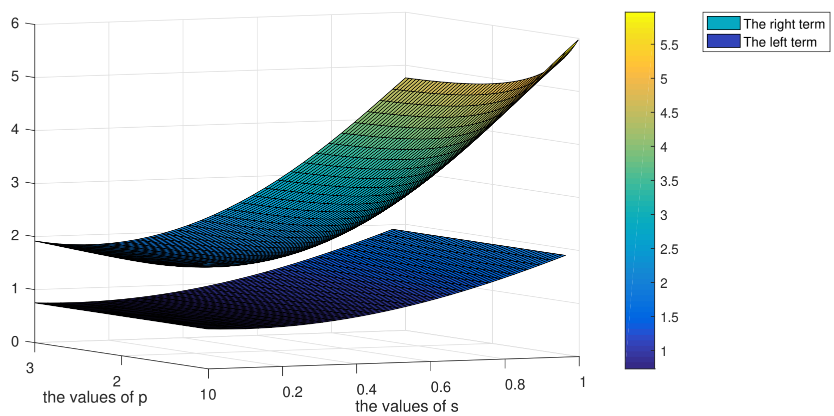

Figure 3. The graph of both sides of the inequality (2.8), which is computed and drawn in MATLAB program, depending on $\mathfrak{s}\in(0,1)$ and $p\in(1,3]$.

3. Conclusion

The goal of this research is to formulate new Hermite-Hadamard-type integral inequalities for twice-differentiable convex mappings through the use of a proportional Caputo hybrid operator. To achieve this, we first present a novel integral identity of the Hermite-Hadamard type associated with the proportional Caputo hybrid operator. Then, we apply convexity, the Hölder inequality, and the power mean inequality to derive various Hermite-Hadamard-type inequalities. Our findings for $\mathfrak{s}\to 1$ which corresponds to some special cases of prior works, are more valuable in this study than in classical calculus. Thus, we hope that our methods and results will motivate readers to further explore this field. Future work may involve exploring similar inequalities for other fractional integrals and generating new Hermite-Hadamard-type inequalities using various convexity approaches.

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Data sharing is not applicable to this paper as no data sets were generated or analysed during the current study.

Conflicts of interest

The authors declare that they have no conflicts of interest.

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