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Three Solutions for a Perturbed Integral Equation with Homogeneous Dirichlet Condition

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Abstract We consider the integral equation $-\mathcal{L}_K^p u = \lambda f(x,u) + \mu g(x,u)$, with homogeneous Dirichlet condition on a bounded Lipschitz domain of $\mathbb{R}^N$ where $\lambda, \mu \in \mathbb{R}$, $p \geq 2$, $s \in]0,1[$, $N > ps$, $f, g : \mathbb{R}^N \longrightarrow \mathbb{R}$ are Carathéodory functions with subcritical growth and $-\mathcal{L}^p$ denotes a class of operators that includes $(-\Delta)_p^s$, the fractional p-Laplacian. Here μg represents a small perturbation of λf. Applying an abstract critical point theorem due to Ricceri, a variational setting developed by Xiang et al. and a Minti-Browder's theorem, we prove the existence of three weak solutions.

Keywords Integral elliptic equation, variational methods, critical points

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1. Introduction

Let's consider a theoretical model for the dynamics of a population living in a habitat $\Omega \subseteq \mathbb{R}^N$ which is an open Lipschitz bounded domain,

$$\partial_t u = \Lambda u + \sigma(x,u), \quad t \geq 0, x \in \Omega. \tag{1.1}$$

Here $u(t,x)$ denotes the population density at time t and position $x \in \Omega$, and the function σ represents the population supply due to births and deaths. The operator Λ, which could be integral or integro-differential, models a diffusion process which is affected by non-local population information coming e.g. from cognitive processes and that, therefore, is very far from considering the individuals as non-living particles interacting in a random way, as it was assumed in the pioneer work [22] in the 1950s.

For the study of biological populations, [24], are of interest the stationary counterparts of equations like (1.1),

$$-\Lambda u = \sigma(x,u), \quad x \in \Omega. \tag{1.2}$$

These time-independent problems have attracted the attention of mathematicians as they appear quite naturally not only in the study of population dynamics but also in continuum mechanics, phase transition phenomena, financial mathematics, game theory, etc.; see e.g. [4, 8, 13, 16, 23]. Particular attention has attracted (1.2),

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for situations where the non-local non-linear diffusion operator Λ equals or contains an integral operator given by

$$\mathcal{L}_K^p u(x) = -2 \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} |u(x) - u(y)|^{p-2}(u(x) - u(y))K(x - y)\, dy, \tag{1.3}$$

where $p > 1$. The function $K : \mathbb{R}^N \setminus \{0\} \longrightarrow \mathbb{R}$ could be interpreted as a *perceptual kernel* (or *detection function*), related to what an individual perceives, and $\mathcal{L}_K^p u(\cdot)$ could be interpreted as a kind of *resource perception function*, which is able to capture information of how the individuals perceived the resources in its habitat, [24]. In this way, the population dynamics modeled with (1.1) would be directly affected by the capacity of perception of the individuals.

We shall assume that the kernel K is positive, even and such that

$$mK \in \mathrm{L}^1(\mathbb{R}^N), \tag{1.4}$$

$$K(x) \geq \theta |x|^{-(N+sp)}, \quad x \in \mathbb{R}^N \setminus \{0\}, \tag{1.5}$$

where $m(x) = \min\{|x|^p, 1\}$ and $\theta > 0$. Observe that, in particular, $\mathcal{L}_K^p$ becomes the fractional p-Laplace operator when $K(x) = |x|^{-(N+ps)}$:

$$-(-\Delta)_p^s u(x) = 2 \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} dy. \tag{1.6}$$

Remark 1.1. To deal with equation (1.2) where $\Lambda = (-\Delta)_p^s$, $0 < s < 1$, a first feasible space to look for weak solutions is the fractional Sobolev space $\mathrm{W}^{s,p}(\Omega)$, [10]. In this context the corresponding embeddings are related to the so-called fractional critical exponent

$$p_s^* = Np/(N - sp).$$

In [25] it was studied a Kirchhoff-type problem with homogeneous Dirichlet condition, that is, equation (1.2) with

$$u(x) = 0, \qquad x \in \mathbb{R}^N \setminus \Omega, \tag{1.7}$$

where

$$\Lambda u = M\Big(\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^p K(x - y)\, dx\, dy\Big)\, \mathcal{L}_K^p u,$$

M is a continuous function and σ is a Carathéodory function verifying Ambrosetti-Rabinowitz condition. When the population supply σ presents a sublinear growth, a direct variational method is applied to obtain solutions, while a mountain-pass solution was found for the case of superlinear population supply.

In [5] and [20] it was studied (1.2) with boundary condition (1.7) for $\Lambda = \mathcal{L}_K^2$. By applying [19, Th.3], three solutions were found in [5] when the population supply has the form

$$\sigma(x, \tau) = \epsilon f(x, \tau) - \lambda g(x, \tau) + \nu h(x, \tau),$$

where $\epsilon, \lambda, \nu \in \mathbb{R}$ and the functions $f, g, h \in \mathrm{C}(\Omega \times \mathbb{R})$ present subcritical growth. On the other hand, in [20] it's found a mountain-pass solution when, for some $a_1, a_2, r > 0$, $\mu > 2$ and $q \in]2, 2_s^*[$, the function σ verifies

$$|\sigma(x, \tau)| \leq a_1 + a_2 |\tau|^{q-1}, \quad \text{for all } \tau \in \mathbb{R} \text{ and a.e. } x \in \Omega;$$

$$\sigma(x,\tau)/|\tau| \longrightarrow 0, \quad \text{as } |\tau| \longrightarrow 0, \text{ uniformly in } x \in \Omega;$$
$$0 < \mu\Sigma(x,\tau) \leq \tau\,\sigma(x,\tau), \quad \text{for all } |\tau| \geq r \text{ and a.e. } x \in \Omega,$$

where $\Sigma(x,\tau) = \int_0^\tau \sigma(x,z)dz$. In this case, the functions are of the form $\sigma(x,\tau) = a(x)|\tau|^{q-2}s$, where $a \in \mathrm{L}^\infty(\Omega)$.

Problem (1.2)-(1.7) with $\Lambda = (-\Delta)_p^s$ has been considered in several works. The semilinear case, $p = 2$, helps to model phenomena in continuum mechanics, phase transition, population dynamics and game theory; e.g., this was the case of [14] where, under several assumptions for σ, finite multiplicity results were obtained by using Morse theory arguments. The quasilinear case, $p > 1$, is also important as it appears in applications; e.g., in the context of quantum mechanics it was studied in [3] where a Ljusternik-Schnirelmann scheme was applied to prove multiplicity and semi-classical concentration (when Planck's constant is considered as a parameter that tends to zero) of positive solutions for a non-local Schrödinger equation.

In this paper we study (1.2)-(1.7) where $\Lambda = \mathcal{L}_K^p$ and the population supply is bi-parameterized,

$$\sigma(x,\tau) = \lambda f(x,\tau) + \mu g(x,\tau),$$

$\lambda, \mu \in \mathbb{R}$. Here λf models the principal component of the population supply while μg stands for a small perturbation of it. Concretely, we consider the problem

$$\begin{cases} -\mathcal{L}_K^p u = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P}$$

where

$$0 < s < 1, \qquad 2 \leq p < \frac{N}{s}, \tag{1.8}$$

and $f, g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ shall be Carathéodory functions with subcritical growth: for some $a_1, a_2, a_3, a_4 \geq 0$ and $\beta, \Upsilon \in]1, p_s^*[$,

$$|f(x,\tau)| \leq a_1 + a_2|\tau|^{\beta-1}, \tag{1.9}$$
$$|g(x,\tau)| \leq a_3 + a_4|\tau|^{\Upsilon-1}, \tag{1.10}$$

for all $\tau \in \mathbb{R}$ and a.e. $x \in \Omega$. Let's denote

$$F(x,\tau) = \int_0^\tau f(x,z)dz \quad \text{and} \quad G(x,\tau) = \int_0^\tau g(x,z)dz,$$

for all $\tau \in \mathbb{R}$ and a.e. $x \in \Omega$.

Before presenting our main result, let's add some necessary words concerning problem (P) in the particular situation of $\Lambda = \mathcal{L}_K^p = (-\Delta)_p^s$.

a) In [28] and [27], by using variational approaches and Morse theory, the authors find that problem (P) has at least three non-trivial solutions for the case of $\mu = 0$, (1.8) and Ω having a $C^{1,1}$ boundary. In [28], for $\lambda > 0$ small enough and f having $(p-1)$-sublinear growth near zero, the authors show that two of the solutions have fixed sign and the remaining one is nodal. In [27], for $\lambda = 1$ and f not verifying the standard Ambrosetti-Rabinowitz condition, the authors show that one of the solutions is positive and one is negative.

b) In [15], (P) was considered for the case of $f(x, s) = |s|^{q-2}s$, $1 < q < p$, $\mu = 1$, (1.8), g verifying (1.10) with $p < \Upsilon < p_s^*$, and Ω having a $C^{1,1}$ boundary. For $\lambda > 0$ small enough, by using critical point theory and Morse theory, the authors prove the existence of at least four non-trivial solutions: two positive, one negative and one nodal.

c) In [12], (P) was considered for the case of $\mu = 0$, $\lambda = 1$, (1.8), Ω has $C^{1,1}$ boundary, and the function f

   i) presents subcritical growth with non-resonance above the first eigenvalue of $(-\Delta)_p^s$ so that the energy functional associated to (P) is noncoercive,

   ii) has $(p-1)$-sublinear growth near zero, and

   iii) vanishes at three points and verifies a reverse Ambrosetti-Rabinowitz condition.

   By using critical point theory and Morse theory, five non-trivial solutions of (P) are found: two positive, two negative and one nodal.

Our main result is the following multiplicity theorem.

Theorem 1.1. *Let $p \geq 2$. Assume (1.4), (1.5), (1.9) and that there exist $c, d > 0$ and $\gamma \in]1, p[$ such that*

(F1) $F(x, \tau) > 0$, for all $\tau \in [0, c]$ and a.e. $x \in \Omega$;

(F2) there exists $\alpha \in]p, p_s^*[$, such that

$$\limsup_{\tau \longrightarrow 0} \frac{\sup_{x\in\Omega} F(x, \tau)}{|\tau|^\alpha} < +\infty;$$

(F3) $|F(x, \tau)| \leq d(1 + |\tau|^\gamma)$, for all $\tau \in \mathbb{R}$ and a.e. $x \in \Omega$.

Then, there exist $\rho > 0$ and an open interval $\Lambda \subseteq]0, +\infty[$ with the following properties:

1. *For every $\lambda \in \Lambda$ and every Carathéodory function g verifying condition (1.10), there exists $\mu \geq 0$ such that (P) has three weak solutions.*
2. *Moreover, there exists $\delta = \delta(\lambda, g) > 0$ such that, for each $\mu \in [0, \delta]$, problem (P) has at least three weak solutions whose norms in W_0 are less than ρ.*

The space W_0, which appears in Theorem 1.1, is similar to the fractional Sobolev space $W^{s,p}(\Omega)$ and was first introduced in [25]; see (2.1) below. Observe that in Theorem 1.1 the only condition on the perturbation function g is (1.10). Then, we extend the results of [11], where the case of $p = 2$ was studied.

By a weak solution of (P), we mean a function $u \in W_0$ which satisfies

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} |u(x) - u(y)|^{p-2}(u(x) - u(y))(h(x) - h(y))K(x - y)\,dx\,dy$$
$$= \mu \int_\Omega f(x, u(x))h(x)\,dx + \lambda \int_\Omega g(x, u(x))h(x)\,dx, \tag{1.11}$$

for every $h \in W_0$.

The rest of this work is organized as follows. In Section 2.1, we introduce the space W_0 and recall some of its properties including Sobolev-like embeddings. In Section 2.2, we introduce our main tools: an abstract critical point theorem due to Ricceri, an auxiliary lemma and a Minty-Browder's theorem. In Section 3, we prove Theorem 1.1 and comment on some of its consequences.

2. Preliminaries

In this section, we introduce the variational setting and the main tools we work with.

2.1. The function space W_0

Let's recall that $0 < s < 1 < p < +\infty$, and $\Omega \subseteq \mathbb{R}^N$ is an open Lipschitz bounded domain.

The normed space W contains all the measurable functions $u : \mathbb{R}^N \longrightarrow \mathbb{R}$ such that the restriction to Ω belongs to $\mathrm{L}^p(\Omega)$ and

$$\|u\|_W = \|u\|_{\mathrm{L}^p(\Omega)} + \left(\int_Q |u(x) - u(y)|^p K(x-y)\,dx\,dy\right)^{1/p} < +\infty,$$

where $Q = (\mathbb{R}^N \times \mathbb{R}^N) \setminus (\Omega^c \times \Omega^c)$. It's clear that

$$W_0 = \{u \in W \;/\; u(x) = 0,\ \text{for a.e. } x \in \mathbb{R}^N \setminus \Omega\} \tag{2.1}$$

is a closed linear subspace of W which encodes the null Dirichlet boundary condition. Observe that $\mathrm{C}_0^\infty(\Omega) \subseteq W_0$.

On W_0 a norm equivalent to $\|\cdot\|_W$ is given by

$$\|u\|_{W_0} = \left(\int_Q |u(x) - u(y)|^p K(x-y)\,dx\,dy\right)^{1/p}.$$

In fact, by [25, Lemma 2.3], there exists $\tilde{C} = \tilde{C}(N, p, s, \theta, \Omega) > 0$ such that, for every $v \in W_0$,

$$\|v\|_{W_0} \leq \|v\|_W = \|v\|_{\mathrm{L}^p(\Omega)} + \|v\|_{W_0} \leq \tilde{C}^{1/p}\|v\|_{W_0}. \tag{2.2}$$

It's clear that the mapping $\mathcal{F} : (W_0, \|\cdot\|_{W_0}) \longrightarrow \mathrm{L}^p(Q)$, given by

$$\mathcal{F}[u](x,y) = (u(x) - u(y))K^{1/p}(x-y),$$

is a linear isometry. As an important consequence we have the following result. The proof we present here is simpler than the one originally provided in [25].

Theorem 2.1. *W_0 is a uniformly convex, reflexive and separable Banach space.*

Proof. Since W_0 is isomorphic to $\mathcal{F}[W_0]$, it's enough to show that $\mathcal{F}[W_0]$ is closed in $\mathrm{L}^p(Q)$. Let $(u_n)_{n\in\mathbb{N}} \subseteq W_0$ be such that $(\mathcal{F}(u_n))_{n\in\mathbb{N}}$ converges to some $f \in \mathrm{L}^p(Q)$. By (2.2), $(u_n)_{n\in\mathbb{N}}$ is convergent to some $g \in \mathrm{L}^p(\Omega)$. By [6, Th.4.9], up to a subsequence, $u_n(x) \longrightarrow g(x)$ and $\mathcal{F}[u_n](x,y) \longrightarrow f(x,y)$, as $n \longrightarrow +\infty$, for a.e. $x \in \Omega$ and a.e. $(x,y) \in Q$, respectively. Therefore, for a.e. $(x,y) \in Q$,

$$\mathcal{F}[u_n](x,y) \longrightarrow (g(x) - g(y))(K(x-y))^{1/p}, \quad \text{as } n \longrightarrow +\infty,$$

and $f(x,y) = (g(x) - g(y))(K(x-y))^{1/p}$. Therefore, $g \in W_0$ and $f = \mathcal{F}(g)$. □

Remark 2.1. Let's also recall, [25], that the embedding $W_0 \subseteq \mathrm{L}^q(\Omega)$ is continuous for $q \in [1, p_s^*]$ and compact for $q \in [1, p_s^*[$. In particular, given $q \in [1, p_s^*]$, there exists $c_q > 0$ such that, for every $u \in W_0$,

$$\|u\|_{\mathrm{L}^q(\Omega)} \leq c_q\|u\|_{W_0}. \tag{2.3}$$

2.2. Main tools

Here we present the tools that shall be used to prove Theorem 1.1: two results produced by Ricceri, a critical point theorem and an auxiliary proposition, and a Minty-Browder's theorem.

Let's start with the abstract result which is our main tool, [18].

Theorem 2.2. *Let E be a reflexive real Banach space and $I \subseteq \mathbb{R}$ an interval. Let $\Phi, \Psi \in \mathrm{C}^1(E)$ be such that*

a) Φ is bounded and weakly sequentially lower semicontinuous;

b) Φ' has a continuous inverse on E';

c) Ψ has a compact differential;

d) for all $\lambda \in I$ $\Phi + \lambda\Psi$ is coercive, i.e.,

$$\Phi(u) + \lambda\Psi(u) \longrightarrow +\infty, \quad \text{as } \|u\| \longrightarrow +\infty; \tag{2.4}$$

e) there exists $z_0 \in \mathbb{R}$ such that

$$\sup_{\lambda\in I} \inf_{u\in E} (\Phi(u) + \lambda(\Psi(u) + z_0)) < \inf_{u\in E} \sup_{\lambda\in I} (\Phi(u) + \lambda(\Psi(u) + z_0)). \tag{2.5}$$

Then, there is an open interval $\Lambda \subseteq I$ and $\rho > 0$ such that given any $\lambda \in \Lambda$ and any $J \in \mathrm{C}^1(E)$ with compact differential, there is $\delta > 0$ such that, for each $\mu \in]0, \delta]$, the functional $\Phi + \lambda\Psi + \mu J$ has at least three critical points whose norms are less than ρ.

To produce Theorem 1.1, we shall apply Theorem 2.2 when the functionals $\Phi, \Psi, J : W_0 \longrightarrow \mathbb{R}$ are given by

$$\Phi(u) = \frac{1}{p}\|u\|^p_{W_0}, \tag{2.6}$$

$$\Psi(u) = -\int_\Omega F(x, u(x))dx, \tag{2.7}$$

$$J(u) = -\int_\Omega G(x, u(x))dx.$$

To get (2.5), the following result will be useful, [17].

Proposition 2.1. *Let X be a non-empty set and $\Phi, \Psi : X \longrightarrow \mathbb{R}$. Assume that there exist $r > 0$ and $u_0, u_1 \in X$ such that $\Phi(u_0) = -\Psi(u_0) = 0$, $\Phi(u_1) > r$ and*

$$\sup_{u\in\Phi^{-1}]-\infty,r]} (-\Psi(u)) = z_1 < z_2 = r\frac{-\Psi(u_1)}{\Phi(u_1)}. \tag{2.8}$$

Then, for each $z \in]z_1, z_2[$,

$$\sup_{\lambda\geq 0} \inf_{u\in E} (\Phi(u) + \lambda(\Psi(u) + z)) < \inf_{u\in E} \sup_{\lambda\geq 0} (\Phi(u) + \lambda(\Psi(u) + z)).$$

To get point b) in Theorem 2.2, we will use the following result which is just one of the consequences of a Minty-Browder's theorem presented e.g. in [26, Th.26A].

Theorem 2.3. *Let E be a reflexive Banach space and $A : E \longrightarrow E'$ coercive, uniformly monotone and hemicontinuous operator. Then, the inverse operator $A^{-1} : E' \longrightarrow E$ exists and is continuous.*

To finish this section, let's recall the concepts appearing in Theorem 2.3. Given a Banach space E, a mapping $A : E \longrightarrow E'$, is said to be

i) coercive if $\langle Au, u\rangle/\|u\| \longrightarrow +\infty$, as $\|u\| \longrightarrow +\infty$;

ii) uniformly monotone if there exists $a : [0, +\infty[\longrightarrow [0, +\infty[$ unbounded, increasing and continuous such that $a(0) = 0$ and, for every $u, v \in E$,

$$\langle Au - Av, u - v\rangle \geq a(\|u - v\|)\, \|u - v\|;$$

iii) hemicontinuous if for every $u, v, w \in E$, the function $]0, 1[\ni t \longmapsto \langle A(u + tv), w\rangle$ is continuous.

3. Multiplicity of solutions

Let's start with the following result.

Lemma 3.1. *Under the hypotheses of Theorem 1.1, the functional Φ given by* (2.6) *is bounded, of class C^1, weakly sequentially lower semicontinuous, and $\Phi' : W_0 \longrightarrow W_0'$ has a continuous inverse.*

Proof. By the form (2.6), it is clear that Φ is weakly sequentially lower semicontinuous and bounded.
a) Let us prove that Φ is Gateaux differentiable. Let $u, \eta \in W_0$, $(x, y) \in Q$ and $\xi \in [-1, 1]$. Then,

$$\begin{aligned} I(x,y,\xi) =& \left|\frac{\partial}{\partial \xi}|u(x) - u(y) + \xi(h(x) - h(y))|^p K(x-y)\right| \\ =& \big||u(x) - u(y) + \xi(h(x) - h(y))|^{p-2}(u(x) - u(y) \\ &+ \xi(h(x) - h(y)))(h(x) - h(y))\big| \cdot K(x-y) \\ \leq& (|u(x) - u(y)| + |\xi||h(x) - h(y)|)^{p-1}|h(x) - h(y)| \cdot K(x-y) \\ \leq& (|u(x) - u(y)| + |h(x) - h(y)|)^{p-1}|h(x) - h(y)| \cdot K(x-y). \end{aligned}$$

By Hölder's inequality, it's easily seen that the function appearing in the last expression belongs to $\mathrm{L}^1(Q)$. So, by Leibniz's integral rule, we have that

$$\begin{aligned} \partial_h \Phi(u) &= \frac{d}{d\xi}\Phi(u + \xi h)\bigg|_{\xi=0} \\ &= \int_Q |u(x) - u(y)|^{p-2}(u(x) - u(y))(h(x) - h(y))K(x-y)\, dx\, dy. \end{aligned}$$

Let's show that the linear mapping $W_0 \ni w \longmapsto \partial_w \Phi(u) \in \mathbb{R}$ is continuous. Let $w \in W_0$. Then, by Höder's inequality,

$$\begin{aligned} |\partial_h \Phi(u)| &\leq \int_Q |u(x) - u(y)|^{p-1} K^{1/p'}(x-y)|h(x) - h(y)|K^{1/p}(x-y)dxdy \\ &\leq \|u\|_{W_0}\|h\|_{W_0}. \end{aligned} \tag{3.1}$$

Then, Φ is Gateaux differentiable and $\langle \Phi'_G(u), h\rangle = \partial_h \Phi(u)$, for all $u, h \in W_0$.
b) Let's prove that Φ is of class C^1. Let $(u_n)_{n\in\mathbb{N}} \subseteq W_0$ and $u \in W_0$ such that

$$u_n \longrightarrow u, \quad \text{as } n \longrightarrow +\infty, \text{ in } W_0. \tag{3.2}$$

By (2.3) and [6, Th.4.9], up to a subsequence, $u_n(x) \longrightarrow u(x)$, as $n \longrightarrow +\infty$, for a.e. $x \in \mathbb{R}^N$. Then, the sequence of functions given by

$$Q \ni (x,y) \longmapsto |u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(K(x-y))^{1/p'} \in \mathbb{R}, \tag{3.3}$$

is bounded in $L^{p'}(Q)$, and converges a.e. in $\mathbb{R}^N$ to the function

$$Q \ni (x,y) \longmapsto |u(x) - u(y)|^{p-2}(u(x) - u(y))(K(x-y))^{1/p'} \in \mathbb{R}.$$

Thus, by (3.2) and Brezis-Lieb's lemma, [7], we get

$$\begin{aligned} &\lim_{n\longrightarrow+\infty} \int_Q \big[|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) \\ &-|u(x) - u(y)|^{p-2}(u(x) - u(y))\big]\, K(x-y)dxdy \\ =&\lim_{n\longrightarrow+\infty} \int_Q \left[|u_n(x) - u_n(y)|^p K(x-y) - |u(x) - u(y)|^p K(x-y)\right] dxdy \\ =&0. \end{aligned} \tag{3.4}$$

By (3.4), Hölder's inequality and working as it was to get (3.1), we obtain, up to a subsequence, that

$$\begin{aligned} \|\Phi'_G(u_n) - \Phi'_G(u)\| &= \sup_{\|v\|_{W_0}\le 1} |\langle \Phi'_G(u_n) - \Phi'_G(u), v\rangle| \\ \le \sup_{\|v\|_{W_0}\le 1} &\bigg[\bigg(\int_Q \big||u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) \\ &-|u(x) - u(y)|^{p-2}(u(x) - u(y))\big|\, K(x-y)dxdy\bigg) \|v\|_{W_0}\bigg] \longrightarrow 0, \end{aligned}$$

as $n \longrightarrow +\infty$. By the arbitrariness of u, Φ'_G is continuous. Therefore (see e.g. [1, pp.1]), Φ is Fréchet differentiable, $\Phi' = \Phi'_G$ and, actually, of class C^1.
c) Let's prove that $\Phi' : W_0 \longrightarrow W_0'$ has a continuous inverse. Since

$$\langle \Phi'(u), u\rangle / \|u\|_{W_0} = \|u\|_{W_0}^{p-1} \longrightarrow +\infty, \quad \text{as } \|u\|_{W_0} \longrightarrow +\infty,$$

Φ' is coercive. Since $p \ge 2$, by point (2.2) in [21], there is some $\check{c}_p > 0$ such that, for $u, v \in W_0$,

$$\begin{aligned} &\langle \Phi'(u) - \Phi'(v), u - v\rangle \\ =&\int_Q [|u(x) - u(y)|^{p-2}(u(x) - u(y)) \\ &- |v(x) - v(y)|^{p-2}(v(x) - v(y))(u(x) - u(y) - v(x) + v(y))]K(x-y)\, dx\, dy \\ \ge&\check{c}_p \int_Q |u(x) - u(y) - v(x) + v(y)|^p K(x-y)\, dx\, dy = \check{c}_p \|u - v\|_{W_0}^p, \end{aligned}$$

so that Φ' is uniformly monotone. Therefore, by Theorem 2.3, Φ' has a continuous inverse.

□

Lemma 3.2. *Assume the hypotheses of Theorem 1.1. Then, the functional* Ψ *given by* (2.7) *is of class* C^1 *and has compact differential.*

Proof. a) The functional Ψ is well-defined. In fact, by (1.9) and (2.3),

$$\begin{aligned}\int_\Omega |F(x,u(x))|\,dx &\le \int_\Omega \int_0^{|u|} [a_1 + a_2|t|^{\beta-1}]\,dt\,dx \\ &= \int_\Omega \left[a_1|u(x)| + \frac{a_2}{\beta}|u(x)|^\beta\right] dx \le a_1 c_1 \|u\|_{W_0} + \frac{a_2}{\beta} c_\beta^\beta \|u\|_{W_0}^\beta.\end{aligned}$$

b) Let's show that Ψ is Gateaux differentiable. Let $u, h \in W_0$, $x \in \Omega$ and $\xi \in [-1,1]$. We have, by (1.9), that

$$\begin{aligned}\left|\frac{\partial}{\partial \xi} F(x, u(x) + \xi h(x))\right| &= \left| f(x, u(x) + \xi h(x)) h(x)\right| \\ &\le (a_1 + a_2|u(x) + \xi h(x)|^{\beta-1})|h(x)| \\ &\le (a_1 + a_2(|u(x)| + |h(x)|)^{\beta-1})|h(x)|.\end{aligned}$$

By Hölder's inequality and (2.3), the function given by the last expression belongs to $\mathrm{L}^1(\Omega)$. Then, by the Leibniz integral rule, we get

$$\partial_h \Psi(u) = \frac{d}{d\xi}\Psi(u + \xi h)\bigg|_{\xi=0} = -\int_\Omega f(x,u(x))h(x)\,dx.$$

Let's show that the linear mapping $W_0 \ni w \longmapsto \partial_w \Psi(u)$ is continuous. Let $w \in W_0$. By (1.9), (2.3) and Hölder's inequality, we get

$$\begin{aligned}|\partial_w \Psi(u)| &\le \int_\Omega |f(x,u(x))|\,|w(x)|\,dx \le a_1\,\|w\|_{\mathrm{L}^1(\Omega)} + a_2 \int_\Omega |u(x)|^{\beta-1}|w(x)|dx \\ &\le a_1\,\|w\|_{\mathrm{L}^1(\Omega)} + a_2\,\|u\|_{\mathrm{L}^\beta(\Omega)}^{\beta-1}\,\|w\|_{\mathrm{L}^\beta(\Omega)} \le (a_1 C_1 + a_2 C_\beta^\beta \|u\|_{W_0}^{\beta-1})\|w\|_{W_0}.\end{aligned}$$

By the arbitrariness of w, Ψ is Gateaux differentiable and $\langle \Psi'_G(u), h\rangle = \partial_h \Psi(u)$, for all $u, h \in W_0$.

c) Let's show that Ψ is of class C^1. Let $(u_n)_{n\in\mathbb{N}} \subseteq W_0$ and $u \in W_0$ such that

$$u_n \longrightarrow u, \quad \text{as } n \longrightarrow +\infty, \text{ in } W_0.$$

By (2.3), $u_n \longrightarrow u$, as $n \longrightarrow +\infty$, in $\mathrm{L}^\beta(\Omega)$. Therefore, by Hölder's inequality, (2.3) and [2, Th.2.2], we get, as $n \longrightarrow +\infty$, that

$$\begin{aligned}|\Psi'(u_n) - \Psi'(u)\| &= \sup_{\|\varphi\|_{W_0}\le 1} \left|\int_\Omega (f(x,u_n) - f(x,u))\varphi(x)dx\right| \\ &\le \sup_{\|\varphi\|_{W_0}\le 1} \|f(\cdot,u_n(\cdot)) - f(\cdot,u(\cdot))\|_{L^{\beta'}(\Omega)}\|\varphi\|_{L^\beta(\Omega)} \\ &\le c_\beta \|f(\cdot,u_n(\cdot)) - f(\cdot,u(\cdot))\|_{L^{\beta'}(\Omega)} \longrightarrow 0. \end{aligned} \tag{3.5}$$

d) Let's show that Ψ' is compact. Let $(u_n)_{n\in\mathbb{N}} \subseteq W_0$ be bounded. By Remark 2.1, up to a subsequence, there is $\tilde{u} \in \mathrm{L}^\beta(\Omega)$ such that $u_n \longrightarrow \tilde{u}$, as $n \longrightarrow +\infty$, in $\mathrm{L}^\beta(\Omega)$, up to a subsequence. By [2, Th.2.2] and taking $u = u_m$, $m \in \mathbb{N}$, in (3.5), it follows that $(\Psi'(u_n))_{n\in\mathbb{N}} \subseteq W_0'$ is a Cauchy sequence as well as convergent. Therefore, Ψ' is compact. □

Now we present the proof of our main result.

Proof. [Proof of Theorem 1.1] It is clear that any critical point of the functional $H = \Phi + \lambda\Psi + \mu J$ is a weak solution of (P). Observe that Theorem 2.1 guarantees the reflexivity of W_0. Thanks to Lemmas 3.1 and 3.2, the functionals Φ and Ψ satisfy the initial hypotheses of Theorem 2.2.

a) Let's show that $\Phi + \lambda\Psi$ is coercive. Let $u \in W_0$ and $\lambda \geq 0$. We have, by (F3) and (2.3), that

$$\begin{aligned}\lambda\Psi(u) &\geq -\lambda d \int_\Omega (1 + |u(x)|^\gamma)dx = -\lambda d\left(|\Omega| + \|u\|^\gamma_{L^\gamma(\Omega)}\right)\\ &\geq -\max\{\lambda d|\Omega|, c_\gamma^\gamma\}(1 + \|u\|^\gamma_{W_0}) = -C_{\lambda,\gamma}(1 + \|u\|^\gamma_{W_0}),\end{aligned}$$

where $C_{\lambda,\gamma} = \max\{\lambda d|\Omega|, c_\gamma^\gamma\}$. Then, since $\gamma < p$, we have, for $\lambda > 0$, that

$$\Phi(u) + \lambda\Psi(u) \geq \frac{1}{p}\|u\|^p_{W_0} - C_{\lambda,\gamma}(1 + \|u\|^\gamma_{W_0}) \longrightarrow +\infty,$$

as $\|u\|_{W_0} \longrightarrow +\infty$, so that (2.4) holds.

b) Let's prove (2.5), i.e.,

$$\sup_{\lambda\in I}\inf_{u\in E}(\Phi(u) + \lambda(\Psi(u) + z_0)) < \inf_{u\in E}\sup_{\lambda\in I}(\Phi(u) + \lambda(\Psi(u) + z_0)),$$

for some $z_0 \in \mathbb{R}$, by showing that the conditions of Proposition 2.1 hold. Observe that $\Phi(0) = -\Psi(0) = 0$. Then we will get (2.5) by showing that there exist $r > 0$ and $u_1 \in W_0$ such that $\Phi(u_1) > r$ and (2.8) hold.

By (F2), there exist $\eta \in]0, 1]$ and $C_1 > 0$ such that

$$F(x, \tau) < C_1|\tau|^\alpha, \quad \text{for every } \tau \in [-\eta, \eta] \text{ and a.e. } x \in \Omega.$$

Then, by (F3), we have, for a.e. $x \in \Omega$,

$$F(x, \tau) \leq \begin{cases} C_1|\tau|^\alpha, & \text{if } \tau \in [0, \eta[,\\ d(1 + |\tau|^\gamma) \leq M_1|\tau|^\alpha, & \text{if } \tau \in [\eta, 1],\\ d(1 + |\tau|^\gamma) \leq d(1 + |\tau|^\alpha) \leq 2d|\tau|^\alpha, & \text{if } t \in]1, +\infty[,\end{cases}$$

where $M_1 = \sup_{\tau\in[\eta,1]} d(1 + |\tau|^\gamma)|t|^{-\alpha}$. Therefore, we can pick $M > 0$ such that

$$F(x, \tau) < M|\tau|^\alpha, \quad \text{for every } t \in \mathbb{R} \text{ and a.e. } x \in \Omega.$$

Then, given $r > 0$, we have, by (2.3), that

$$-\Psi(u) = \int_\Omega F(x, u(x))\, dx \leq M\, \|u\|_{\mathrm{L}^\alpha(\Omega)} \leq c_\alpha^\alpha \|u\|^\alpha_{W_0} \leq c_\alpha^\alpha p^{\alpha/p} r^{\alpha/p},$$

for every $u \in \Phi^{-1}(] - \infty, r])$. Hence, since $\alpha > p$, it follows that

$$\lim_{r\downarrow 0} r^{-1} \sup_{\|u\|^p_{W_0}\leq pr} (-\Psi(u)) = 0. \tag{3.6}$$

By (F1), there is $c > 0$ such that $F(x, \tau) > 0$, for all $\tau \in [0, c]$ and a.e. $x \in \Omega$. Now let's pick a positive function $u_1 \in \mathrm{C}_0^\infty(\Omega) \subseteq W_0$ such that

$$\max_{x \in \overline{\Omega}} u_1(x) \leq c.$$

Then, $\Phi(u_1) > 0$ and, in view of (F1), it also holds $-\Psi(u_1) > 0$. Therefore, by (3.6), we can find $r \in]0, \Phi(u_1)[$ such that

$$\sup_{u \in \Phi^{-1}(]-\infty, r])} (-\Psi(u)) = \sup_{\|u\|_{W_0}^p \leq pr} (-\Psi(u)) < r \frac{-\Psi(u_1)}{\Phi(u_1)}.$$

c) Observe that the argument of Lemma 3.2 also shows that J is of class C^1 with compact differential. We conclude by Theorem 2.2. □

Let's now consider the problem

$$\begin{cases} (-\Delta)_p^s u = \lambda f(u) + \mu(1 + |u|^\beta), & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P'}$$

where $\beta \in]0, p_s^* - 1[$. As a consequence of Theorem 1.1, we have, by considering the kernel given by $K(x) = |x|^{-(N+sp)}$, the following extension of [11, Cor.3.4].

Corollary 3.1. *Let $s \in]0, 1[$, $N > ps$, $p \geq 2$ and $\Omega \subseteq \mathbb{R}^N$ be a bounded Lipschitz domain. Assume that $f \in \mathrm{C}(\mathbb{R})$ verifies*

$$\sup_{t \in \mathbb{R}} \frac{|f(t)|}{1 + |t|^{\sigma - 1}} < +\infty,$$

for some $\sigma \in]1, p_s^*[$, and, for some $c, d > 0$ and $\gamma \in]1, p[$,

(F1') $F(t) > 0$, for every $t \in [0, c]$;

(F2') there exists $\alpha \in]p, p_s^*[$ such that $\limsup_{t \longrightarrow 0} |t|^{-\alpha} F(t) < +\infty$;

(F3') $|F(t)| \leq d(1 + |t|^\gamma)$, for every $t \in \mathbb{R}$.

Then, there exist $\rho > 0$ and an open interval $\Lambda \subseteq]0, +\infty[$ with the following property. For each $\lambda \in \Lambda$ there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the problem (P') *has at least three weak solutions in $\mathrm{W}^{s,p}(\mathbb{R}^N)$ whose norms in W_0 are less than ρ.*

A simple example of application of Corollary 3.1 corresponds to the function given by

$$f(t) = \begin{cases} 0, & \text{if } t < 0, \\ t^{\alpha - 1}, & \text{if } 0 \leq t \leq 1, \\ t^{\gamma - 1}, & \text{if } t > 1, \end{cases}$$

where $p < \alpha < 3p/2$, $1 < \gamma < p$ and $\beta \in]0, 3(p-1)/2[$. In this case, the three solutions provided by Corollary 3.1 belong to the space $\mathrm{W}^{1/p,p}(\Omega)$.

Remark 3.1. In [23] it was studied the problem

$$\begin{cases} -\mathcal{L}_K^2 u \in \lambda(\partial \tilde{f}(x, u)) + \mu \partial \tilde{g}(x, u)), & \text{in } \Omega \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{Q}$$

where $\tilde{f}, \tilde{g} : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}$ are suitable measurable functions such that, for a.e. $x \in \Omega$, $\tilde{f}(x, \cdot)$ and $\tilde{g}(x, \cdot)$ are locally Lipschitz continuous functions. Here, $\partial \tilde{f}(x, \cdot)$ and $\partial \tilde{g}(x, \cdot)$ denote Clark's generalized subdifferential, [9]. The authors proved the existence of two non-trivial weak solutions for suitable values of the parameters $\lambda, \mu \in \mathbb{R}$. Theorem 1.1 and the smooth version of the above mentioned result are independent, [11].

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