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Properties Around the First Eigenvalue of a Partial Discrete Dirichlet Boundary Value Problem

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Abstract In this article, we study some results around the first eigenvalue $\lambda_{1,m}$ for the partial discrete Dirichlet boundary value problem, like the constant sign of the eigenfunction associated with $\lambda_{1,m}$, the simplicity of $\lambda_{1,m}$ and the sign change of any eigenfunction associated with $\lambda > \lambda_{1,m}$.

Keywords Partial discrete Dirichlet problem, first eigenvalue, simplicity

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1. Introduction

Let λ be a positive parameter and $\mathbb{Z}[N, M]$ represent the discrete interval $\{N, N+1, ..., M\}$, where N and M are integers and $N < M$. We consider the following partial discrete problem

$$(P_\lambda)\begin{cases} -\Delta_1(\varphi_p(\Delta_1 x(i-1,j))) - \Delta_2(\varphi_p(\Delta_2 x(i,j-1))) = \lambda m(i,j)\varphi_p(x(i,j)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (i,j) \in \mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta], \\ x(0,j) = x(\alpha+1,j) = 0, \qquad j \in \mathbb{Z}[1,\beta], \\ x(i,0) = x(i,\beta+1) = 0, \qquad i \in \mathbb{Z}[1,\alpha], \end{cases}$$

where $\alpha, \beta \geq 2$ are fixed positive integers, Δ_1 and Δ_2 denote the forward difference operators defined by $\Delta_1 x(i,j) = x(i+1,j) - x(i,j)$ and $\Delta_2 x(i,j) = x(i,j+1) - x(i,j)$, $m \in M_+ =: \{m : \mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta] \to \mathbb{R}/\exists(i_0,j_0) \in \mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta] : m(i_0,j_0) > 0\}$, φ_p denotes the p-Laplacian operator defined by $\varphi_p(u) = |u|^{p-2}u$ and $1 < p < \infty$.

Recent mathematical literature has given difference equations a great deal of attention. Research on these equations can be found at the intersection of many mathematics disciplines, including numerical analysis and nonlinear differential equations. Moreover, they are strongly motivated by their applicability to various fields of research, such as artificial or biological neural networks, mechanical engineering, control systems, and more. For studying these equations, using techniques for nonlinear analysis, many authors arrived at a variety of results, such as fixed point

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methods, Brouwer degree, and critical point theory [1–5,9,12,13,17]. Partial difference equations are those difference equations that include two or more variables and are less frequently studied [7,11,14,16]. Partial difference equations have recently been extensively utilized in a variety of fields.

In 2010, Galewski and Orpe [10] demonstrated the existence of solutions to the following problem utilizing critical point theory:

$$(E_\lambda^f)\begin{cases} -\Delta_1^2(x(i-1,j))) - \Delta_2^2(x(i,j-1) = \lambda f((i,j),x(i,j)), \\ \qquad\qquad\qquad\qquad\qquad\qquad (i,j)\in\mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta], \\ x(0,j)=x(\alpha+1,j)=0, \qquad j\in\mathbb{Z}[1,\beta], \\ x(i,0)=x(i,\beta+1)=0, \qquad i\in\mathbb{Z}[1,\alpha], \end{cases}$$

where Δ_1^2 and Δ_2^2 are two second-order forward difference operators given by $\Delta_1^2 x(i,j) = \Delta_1(\Delta_1 x(i,j))$ and $\Delta_2^2 x(i,j) = \Delta_2(\Delta_2 x(i,j))$, respectively.

Heidarkhani and Imbesi [15] in 2015 presented certain conditions to establish multiple solutions for the problem (E_λ^f).

Du and Zhou [8] studied the following problem in 2020:

$$(E_\lambda^{f,p})\begin{cases} -\Delta_1(\varphi_p(\Delta_1 x(i-1,j))) - \Delta_2(\varphi_p(\Delta_2 x(i,j-1))) = \lambda f((i,j),x(i,j)), \\ \qquad\qquad\qquad\qquad\qquad\qquad (i,j)\in\mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta] \\ x(0,j)=x(\alpha+1,j)=0, \qquad j\in\mathbb{Z}[1,\beta], \\ x(i,0)=x(i,\beta+1)=0, \qquad i\in\mathbb{Z}[1,\alpha], \end{cases}$$

where $1<p<\infty$ and $f((i,j),.)\in C(\mathbb{R},\mathbb{R})$, for all $(i,j)\in\mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta]$. By using critical point theory, the authors established the existence of infinitely many solutions for $(E_\lambda^{f,p})$.

Our study focuses on some results concerning the first eigenvalue $\lambda_{1,m}$ of the problem (P_λ). More precisely, in this article, we investigate the simplicity of $\lambda_{1,m}$, the constant sign of the first eigenfunction associated with $\lambda_{1,m}$, the strict monotonicity characteristic concerning the weight and sign change of any eigenfunction associated with $\lambda>\lambda_{1,m}$.

We consider the $\alpha\beta$-dimensional Banach space

$$H=\Big\{x:\mathbb{Z}[0,\alpha+1]\times\mathbb{Z}[0,\beta+1]\longrightarrow\mathbb{R}: x(0,j)=x(\alpha+1,j)=0,\ j\in\mathbb{Z}[1,\beta] \text{ and } \\ x(i,0)=x(i,\beta+1)=0,\ i\in\mathbb{Z}[1,\alpha]\Big\},$$

endowed with the norm

$$\|x\| = \Big(\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}\Delta_1 x(i-1,j)^2 + \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}\Delta_2 x(i,j-1)^2\Big)^{\frac{1}{2}}.$$

This article's last section is structured as follows: Section 2 is devoted to mathematical preliminaries, and statements of the main results are covered. Section 3 contains proof of the main results.