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The Second-Order Melnikov Function and Chaotic Tangles on Toral van der Pol Equation*

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Abstract The van der Pol equation on the torus is considered. This equation contains a heteroclinic cycle consisting of four symmetric heteroclinic orbits. By high order Melnikov method, the periodic forced van der Pol equation is investigated and chaotic dynamics is obtained. An explicit formula of the second-order Melnikov function is derived for the splitting heteroclinic connection. It is used to detect chaotic dynamics when the first-order(classical) Melnikov function is degenerate. By the second-order Melnikov function, we deduce chaotic heteroclinic tangles with rigorously theoretical analysis as well as numerical simulations.

Keywords Van der Pol equation, high order Melnikov method, heteroclinic tangles

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1. Introduction

As we have known, the planar van der Pol equation provides a typical example of relaxation oscillators which possess limit cycles by Poincaré-Bendixson theorem [1]. For the strongly forced van der Pol equation, Levinson [2] and Levi [3] noted that there exists an attracting set involving homoclinic tangles. In the 1960s, Smale observed that horseshoe, which represents chaos in the sense of topology, occurs in all homoclinic tangles [4]. Recently, Refs. [5–7] presented that periodic sinks and Hénon-like attractors are also residing in homoclinic and heteroclinic tangles.

Melnikov's method is one of the few analytical methods available for the detection of chaotic motions [1,8–11]. This method establishes a basic Melnikov function $M_0(t_0)$ to check the existence of transverse homoclinic points by simple zeros. However, there exist equations such that $M_0(t_0)$ is degenerate. In this case we need to consider high order Melnikov functions to analyze the separation between stable and unstable manifolds. Although Ref. [12, 13] proposed a scheme to derive high order Melnikov functions, it is hard for one to apply in the concrete examples. Recently, Refs. [14–17] developed an entire theory of high order Melnikov functions

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for time-periodic equations. In particular, Ref. [14] shows an explicit formula of the second-order Melnikov function which can be computable in applications.

More complicated than homoclinic tangles, heteroclinic tangles create from transversal intersections of stable and unstable manifolds associated with different saddles. For example, if a diffeomorphism possesses n heteroclinic points and forms a transversal n-cycle, then it implies a transversal homoclinic point and chaos arising [18]. This paper studies dynamics of van der Pol equation on torus which admits four heteroclinic solutions to two saddles. We consider periodically forced van der Pol equation. We present, through numerical simulations, chaotic heteroclinic tangles with the first-order Melnikov function and the second-order Melnikov function.

2. High order Melnikov's method

We consider as identical all points whose coordinates differ by 2π, and use $\mathbb{T} = [-\pi, \pi] \times [-\pi, \pi]$ to denote the torus. The forced van der Pol equation on torus $\mathbb{T}$ can be written in the form

$$\begin{aligned} \dot{x} &= \sin y, \\ \dot{y} &= -\sin x + \varepsilon P(x, y, t), \end{aligned} \tag{2.1}$$

where $P(x, y, t)$ is a real analytic bounded function in $(x, y) \in \mathbb{T}$ and T-periodic in t. Suppose ε is a small parameter, so that eq.(2.1) is a perturbation of autonomous equation

$$\begin{aligned} \dot{x} &= \sin y, \\ \dot{y} &= -\sin x, \end{aligned} \tag{2.2}$$

which has four singular points at $O(0, 0)$, $O_1(\pm\pi, 0)$, $O_2(0, \pm\pi)$ and $O_3(\pm\pi, \pm\pi)$. Within the four singular points, O and O_3 are centers while O_1 and O_2 are saddle points. Moreover, there are four heteroclinic orbits connecting O_1 and O_2, forming a heteroclinic cycle. Fig.1 depicts the phase structure on the plane and on the cylinder. The heteroclinic cycle divides the torus into two regions, one surrounding O and the other surrounding O_3. Let ℓ_{12} denote the heteroclinic orbit from $O_1(-\pi, 0)$ to $O_2(0, \pi)$, and ℓ_{21} the heteroclinic orbit from $O_2(0, \pi)$ to $O_1(\pi, 0)$. By symmetry, we denote the other two heteroclinic orbits by $-\ell_{12}$ and $-\ell_{21}$.