Journal of Nonlinear Modeling and Analysis
Volume 8, Number 1, February 2026, 1–8
https://global-sci.com/jnma
DOI:10.12150/jnma.2026.1

New Refinements of Hermite-Hadamard Inequalities for Quantum Integrals

Hüseyin Budak[1,2,†]

Received 30 October 2024; Accepted 25 February 2025

Abstract In this study, we first introduce two functions including quantum integrals. Then, we prove some properties of these mappings such as convexity and monotony. Moreover, by using the newly defined mappings, we prove some refinements of the left hand side of Hermite-Hadamard inequality for left and right quantum integrals.

Keywords Convex functions, Hermite-Hadamard inequality, quantum integrals

MSC(2010) 26A51, 26D10, 26D15

1. Introduction

Quantum calculus is a mathematical framework that is distinct from traditional infinitesimal calculus, as it does not rely on limits or examine calculations with limits. The term "quantum" originates from the Latin word "Quantus," meaning "how much," or "Kvant" in Swedish. There are two main branches of quantum calculus: the q-calculus and the h-calculus, both of which were investigated by P. Cheung and V. Kac in the early last century. While FH Jackson was studying quantum calculus or q-calculus at the same time, this type of calculus had already been solved by Euler and Jacobi. However, quantum calculus, which originated with Euler, has become a significant area of study in mathematics and physics today, providing solutions to previously unsolvable problems, particularly for discontinuous functions. Tariboon and Ntouyas introduced the concepts of quantum calculus on finite intervals and acquired several q-analogues of classical mathematical materials. This has led to numerous new results in the literature regarding quantum analogues of classical mathematical investigations. Additionally, recent research has shown that quantum calculus is a subfield of the more general mathematical area of time scales calculus. Time scales provide a unified framework for investigating dynamic equations on both discrete and continuous domains, making quantum calculus applicable in many fields such as number theory, combinatorics, orthogonal polynomials, simple hypergeometric functions, quantum theory, physics, and relativity theory. For more information, see [3, 5–12, 15] and references cited therein.

[†]the corresponding author.
Email address:hsyn.budak@gmail.com(H. Budak)
[1]Department of Mathematics, Faculty of Science and Arts, Kocaeli University, Kocaeli 41001, Türkiye
[2]Department of Mathematics, Saveetha School of Engineering, SIMATS, Saveetha University, Chennai 602105, Tamil Nadu, India

The Hermite-Hadamard inequality was independently proven by C. Hermite and J. Hadamard. It's one of the most recognized inequalities in the theory of convex functional analysis, which is stated as follows:

Let $f:[a,b]\to\mathbb{R}$ be a convex mapping. Then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int\limits_{a}^{b}f(x)\,dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

If f is concave, both inequalities hold in the reverse direction. Finding many studies in inequality theory, the quantum integral has gone through various searches by researchers to establish the quantum version of the famous Hermite-Hadamard inequality above.

In [2, 4], Alp et al. and Bermudo et al., with the help of the q-derivatives and integrals (defined in Section 2), derive two different versions of q-Hermite-Hadamard inequalities and some estimates. The q-Hermite-Hadamard inequalities are defined as:

Theorem 1.1. *[2, 4] For a convex mapping $f:[a,b]\to\mathbb{R}$, the following inequalities hold:*

$$f\left(\frac{qa+b}{[2]_q}\right)\leq\frac{1}{b-a}\int_a^b f(x)\ {}_a d_q x\leq\frac{qf(a)+f(b)}{[2]_q}, \tag{1.2}$$

$$f\left(\frac{a+qb}{[2]_q}\right)\leq\frac{1}{b-a}\int_a^b f(x)\ {}^b d_q x\leq\frac{f(a)+qf(b)}{[2]_q}. \tag{1.3}$$

Remark 1.1. It is very easy to observe that by adding (1.2) and (1.3), we have the following q-Hermite-Hadamard inequality (see, [4]):

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq\frac{1}{2(b-a)}\left[\int_a^b f(x)\ {}_a d_q x+\int_a^b f(x)\ {}^b d_q x\right] \\ &\leq\frac{f(a)+f(b)}{2}. \end{aligned} \tag{1.4}$$

Hereabout, Ali et al. [1] and Sitthiwirattham et al. [13] present the following two different and new versions of q-Hermite-Hadamard type inequalities:

Theorem 1.2. *[1, 13] For a convex mapping $f:[a,b]\to\mathbb{R}$, the following inequalities hold:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \\ &\leq\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\ {}^{\frac{a+b}{2}}d_q x+\int_{\frac{a+b}{2}}^b f(x)\ {}_{\frac{a+b}{2}}d_q x\right] \\ &\leq\frac{f(a)+f(b)}{2}, \end{aligned} \tag{1.5}$$

$$f\left(\frac{a+b}{2}\right) \tag{1.6}$$

$$\leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\ {}_a d_q x + \int_{\frac{a+b}{2}}^b f(x)\ {}^b d_q x\right]$$
$$\leq \frac{f(a)+f(b)}{2}.$$

Remark 1.2. If we allow limit as $q \to 1^-$ in (1.2)-(1.6), then the inequalities reduce to classical Hermite-Hadamard inequality (1.1).

The main goal of the paper is to define two functions including quantum integrals. Then we prove the convexity and monotony of these functions. With the help of the newly presented functions, we also acquire some improvement of the left-hand sides of the inequalities of Hermite-Hadamard type inequalities for left and right quantum integrals, (1.2) and (1.3), respectively.

2. Preliminaries of q-calculus and some inequalities

We shall recall some basics of quantum calculus in this section. For the sake of brevity, let $q \in (0,1)$ and we use the following notation (see, [9]):

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \dots + q^{n-1}.$$

Definition 2.1. [14] The left or q_a-derivative of $f : [a,b] \to \mathbb{R}$ at $x \in [a,b]$ is expressed as:

$$ {}_a D_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)},\ x \neq a. \tag{2.1}$$

Definition 2.2. [4] The right or q^b-derivative of $f : [a,b] \to \mathbb{R}$ at $x \in [a,b]$ is expressed as:

$$ {}^b D_q f(x) = \frac{f(qx + (1-q)b) - f(x)}{(1-q)(b-x)},\ x \neq b.$$

Definition 2.3. [14] The left or q_a-integral of $f : [a,b] \to \mathbb{R}$ for $x \in [a,b]$ is defined as:

$$\int_a^x f(t)\ {}_a d_q t = (1-q)(b-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a).$$

Definition 2.4. [4] The right or q^b-integral of $f : [a,b] \to \mathbb{R}$ for $x \in [a,b]$ is defined as:

$$\int_x^b f(t)\ {}^b d_q t = (1-q)(b-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b).$$

3. Main results

In this section, we will define two functions including quantum integrals. We will prove how these functions improve the Hermite-Hadamard inequality.

Theorem 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function and let $\Psi:[0,1]\to\mathbb{R}$ be a function defined by*

$$\Psi(t)=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{qa+b}{[2]_q}\right)\ {}_ad_qx.$$

Then we have:

1. Ψ is convex on $[0,1]$.
2. We have the following inequality:
$$f\left(\frac{qa+b}{[2]_q}\right)\leq\Psi(t)\leq\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx. \tag{3.1}$$
3. Ψ is monotonically increasing on $[0,1]$.

Proof. 1. Let $t,s\in[0,1]$ and $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$. Then by using the convexity of f, we derive

$$\begin{aligned}
&\Psi(\alpha t+\beta s)\\
&=\frac{1}{b-a}\int_a^b f\left((\alpha t+\beta s)x+(1-(\alpha t+\beta s))\frac{qa+b}{[2]_q}\right)\ {}_ad_qx\\
&=\frac{1}{b-a}\int_a^b f\left(\alpha\left(tx+(1-t)\frac{qa+b}{[2]_q}\right)+\beta\left(sx+(1-s)\frac{qa+b}{[2]_q}\right)\right)\ {}_ad_qx\\
&\leq\frac{1}{b-a}\int_a^b\left[\alpha f\left(tx+(1-t)\frac{qa+b}{[2]_q}\right)+\beta f\left(sx+(1-s)\frac{qa+b}{[2]_q}\right)\right]\ {}_ad_qx\\
&=\alpha\Psi(t)+\beta\Psi(s).
\end{aligned}$$

Hence Ψ is convex on $[0,1]$.

2. Since $\sum_{n=0}^{\infty}(1-q)q^n=1$, by using Jensen inequality, we establish

$$\begin{aligned}
\Psi(t)&=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{qa+b}{[2]_q}\right)\ {}_ad_qx\\
&=(1-q)\sum_{n=0}^{\infty}q^nf\left(t\left(q^nb+(1-q^n)a\right)+(1-t)\frac{qa+b}{[2]_q}\right)\\
&\geq f\left(\sum_{n=0}^{\infty}(1-q)q^n\left(t\left(q^nb+(1-q^n)a\right)+(1-t)\frac{qa+b}{[2]_q}\right)\right)\\
&=f\left((1-q)\left(t\left(\frac{b}{1-q^2}+\frac{qa}{1-q^2}\right)+(1-t)\frac{qa+b}{[2]_q}\right)\right)\\
&=f\left(\frac{qa+b}{[2]_q}\right),
\end{aligned}$$

which gives the first inequality in (3.1). For the the proof of second inequality, by using convexity of f, we get

$$\Psi(t)=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{qa+b}{[2]_q}\right)\ {}_ad_qx$$

$$\leq \frac{1}{b-a}\int_a^b \left[tf\left(x\right)+\left(1-t\right)f\left(\frac{qa+b}{\left[2\right]_q}\right)\right] \ _ad_qx$$
$$= \frac{t}{b-a}\int_a^b f\left(x\right) \ _ad_qx+\left(1-t\right)f\left(\frac{qa+b}{\left[2\right]_q}\right) := w(t).$$

It is clear from the inequality (1.2) that w is monotonically increasing on $[0,1]$. Therefore we have

$$\Psi(t) \leq w(t) \leq w(1) = \frac{1}{b-a}\int_a^b f\left(x\right) \ _ad_qx.$$

This finishes the proof of the inequality (3.1).

3. Since Ψ is convex on $[0,1]$, for $t_1, t_2 \in [0,1]$ with $t_2 > t_1$, we obtain

$$\frac{\Psi(t_2)-\Psi(t_1)}{t_2-t_1} \geq \frac{\Psi(t_1)-\Psi(0)}{t_1-0} = \frac{\Psi(t_1)-f\left(\frac{qa+b}{[2]_q}\right)}{t_1}.$$

By the first inequality in (3.1), we have $\Psi(t_1) \geq f\left(\frac{qa+b}{[2]_q}\right)$, so we get

$$\frac{\Psi(t_2)-\Psi(t_1)}{t_2-t_1} \geq 0.$$

That is, $\Psi(t_2) \geq \Psi(t_1)$. This gives that Ψ is monotonically increasing on $[0,1]$. □

Theorem 3.2. *Let $f : [a,b] \to \mathbb{R}$ be a convex function and let $\Upsilon : [0,1] \to \mathbb{R}$ be a function defined by*

$$\Upsilon(t) = \frac{1}{b-a}\int_a^b f\left(tx+\left(1-t\right)\frac{a+qb}{\left[2\right]_q}\right) \ ^bd_qx.$$

Then we have:

1. *Υ is convex on $[0,1]$.*
2. *We have the following inequality:*

$$f\left(\frac{a+qb}{\left[2\right]_q}\right) \leq \Upsilon(t) \leq \frac{1}{b-a}\int_a^b f\left(x\right) \ ^bd_qx. \tag{3.2}$$

3. *Υ is monotonically increasing on $[0,1]$.*

Proof. 1. Let us consider $t, s \in [0,1]$ and $\alpha, \beta \in [0,1]$ with $\alpha+\beta = 1$. Since f is convex on $[a,b]$, we can write

$$\Upsilon(\alpha t+\beta s)$$
$$= \frac{1}{b-a}$$
$$\times \int_a^b f\left(\left(\alpha t+\beta s\right)x+\left(1-\left(\alpha t+\beta s\right)\right)\frac{a+qb}{\left[2\right]_q}\right) \ ^bd_qx$$

$$
\begin{aligned}
&= \frac{1}{b-a} \\
&\quad \times \int_a^b f\left(\alpha\left(tx+(1-t)\tfrac{a+qb}{[2]_q}\right)+\beta\left(sx+(1-s)\tfrac{a+qb}{[2]_q}\right)\right)\ {}^b d_q x \\
&\leq \frac{1}{b-a} \\
&\quad \times \int_a^b \left[\alpha f\left(tx+(1-t)\tfrac{a+qb}{[2]_q}\right)+\beta f\left(sx+(1-s)\tfrac{a+qb}{[2]_q}\right)\right]\ {}^b d_q x \\
&= \alpha\Upsilon(t)+\beta\Upsilon(s).
\end{aligned}
$$

Therefore, Υ is convex on $[0,1]$.

2. Since $\sum_{n=0}^{\infty}(1-q)q^n=1$, by using Jensen inequality, we establish

$$
\begin{aligned}
\Upsilon(t) &= \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+qb}{[2]_q}\right)\ {}^b d_q x \\
&= (1-q)\sum_{n=0}^{\infty} q^n f\left(t\left(q^n a+(1-q^n)\,b\right)+(1-t)\frac{a+qb}{[2]_q}\right) \\
&\geq f\left(\sum_{n=0}^{\infty}(1-q)q^n\left(t\left(q^n a+(1-q^n)\,b\right)+(1-t)\frac{a+qb}{[2]_q}\right)\right) \\
&= f\left((1-q)\left(t\left(\frac{a}{1-q^2}+\frac{qb}{1-q^2}\right)+(1-t)\frac{a+qb}{[2]_q}\right)\right) \\
&= f\left(\frac{a+qb}{[2]_q}\right).
\end{aligned}
$$

This proves the first inequality in (3.2). Since f is convex on $[a,b]$, we have

$$
\begin{aligned}
\Upsilon(t) &= \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+qb}{[2]_q}\right)\ {}^b d_q x \\
&\leq \frac{1}{b-a}\int_a^b \left[tf(x)+(1-t)\,f\left(\frac{a+qb}{[2]_q}\right)\right]\ {}^b d_q x \\
&= \frac{t}{b-a}\int_a^b f(x)\ {}^b d_q x+(1-t)\,f\left(\frac{a+qb}{[2]_q}\right) := g(t).
\end{aligned}
$$

It is clear from the inequality (1.3) that g is monotonically increasing on $[0,1]$. Therefore we have

$$
\Upsilon(t)\leq g(t)\leq g(1)=\frac{1}{b-a}\int_a^b f(x)\ {}^b d_q x.
$$

This finishes the proof of the inequality (3.2).

3. Since Υ is convex on $[0,1]$, for $t_1,t_2\in[0,1]$ with $t_2>t_1$, we obtain

$$
\frac{\Upsilon(t_2)-\Upsilon(t_1)}{t_2-t_1}\geq\frac{\Upsilon(t_1)-\Upsilon(0)}{t_1-0}=\frac{\Upsilon(t_1)-f\left(\frac{a+qb}{[2]_q}\right)}{t_1}.
$$

By the first inequality in (3.2), we have $\Upsilon(t_1) \geq f\left(\frac{a+qb}{[2]_q}\right)$, so we get

$$\frac{\Upsilon(t_2) - \Upsilon(t_1)}{t_2 - t_1} \geq 0.$$

That is, $\Upsilon(t_2) \geq \Upsilon(t_1)$. This gives that Υ is monotonically increasing on $[0,1]$. □

4. Conclusions

In this article, we present an extension of the Hermite-Hadamard inequalities for the quantum integral. In the upcoming directions, researchers can be obtain similar inequalities for right hand sides of the inequalities (1.2) and (1.3) and both sides of the inequalities (1.5) and (1.6).

References

[1] M. A. Ali, H. Budak, M. Fečkan and S. Khan, *A new version of q-Hermite-Hadamard's midpoint and trapezoid type inequalities for convex functions*, Mathematica Slovaca, 73, 2023, 369–386.

[2] N. Alp, M. Z. Sarikaya, M. Kunt and İ. İşcan, *q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions*, J. King Saud Univ. Sci., (30), 2018, 193–203.

[3] M. A. Ali and E. Köbis, *Some new q-Hermite-Hadamard-Mercer inequalities and related estimates in quantum calculus*, J. Nonlinear Var. Anal., 7(1), 2023, 49–66.

[4] S. Bermudo, P. Kórus and J. N. Valdés, *On q-Hermite-Hadamard inequalities for general convex functions*, Acta Math. Hung., 162, 2020, 364–374.

[5] B. Bin-Mohsin, M.Z. Javed, M. U. Awan, A. G. Khan, C. Cesarano, and M. A. Noor, *Exploration of quantum Milne-Mercer-type inequalities with applications*, Symmetry, 15(5), 2023, 1096.

[6] H. Budak, *Some trapezoid and midpoint type inequalities for newly defined quantum integrals*, Proyecciones, 40, 2021, 199-215.

[7] S. I. Butt, M. N. Aftab, H. Nabwey, and S. Etemad, *Some Hermite-Hadamard and midpoint type inequalities in symmetric quantum calculus*, AIMS Mathematics, 9(3), 2024, 5523–5549.

[8] L. Ciurdariu and E. Grecu, *Several quantum Hermite-Hadamard-type integral inequalities for convex functions*, Fractal and Fractional, 7(6), 2023, 463.

[9] V. Kac and P. Cheung, *Quantum calculus*, Springer 2001.

[10] A. Lakhdari, W. Saleh, and B. Meftah, *On Second-Family Radau-type Inequalities*. Journal of Inequalities and Mathematical Analysis, 1(3), 2025, 124–138.

[11] M. A. Noor, K. I. Noor and M. U. Awan, *Some quantum estimates for Hermite-Hadamard inequalities*, Appl. Math. Comput., 251, 2015, 675–679.

[12] W. Sudsutad, S. K. Ntouyas and J. Tariboon, *Quantum integral inequalities for convex functions*, J. Math. Inequal., 9, 2015, 781–793.

[13] T. Sitthiwirattham, M. A. Ali, A. Ali and H. Budak, *A new q-Hermite-Hadamard's inequality and estimates for midpoint type inequalities for convex functions*, Miskolc Mathematical Notes, 24(3), 2023, 1555–1567.

[14] J. Tariboon and S. K. Ntouyas,*Quantum calculus on finite intervals and applications to impulsive difference equations*, Adv. Difference Equ., 282, 2013, 1–19.

[15] M. Vivas-Cortez, M. Z. Javed, M. U. Awan, S. S. Dragomir, and A. M. Zidan, *Properties and applications of symmetric quantum calculus*, Fractal and Fractional, 8(2), 2024, 107.