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Bullen-Type Fractional Integral Inequalities Involving Twice-Differentiable Mappings with Applications

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Abstract In this paper, we first introduce a novel extension of the Bullen-type inequalities using Riemann-Liouville integral operators and establish a generalized fractional integral identity of the Bullen type. Next, we derive integral inequalities related to the Bullen-type inequalities for twice-differentiable mappings, utilizing this novel extended identity. To demonstrate these inequalities, we provide examples along with corresponding graphs. Furthermore, we investigate how these inequalities can be practically applied to mean inequalities, contributing to a deeper comprehension and wider usefulness of these mathematical ideas. Finally, the inequalities given in this work extend several known inequalities in the literature, which is the main advantage of the newly found inequalities.

Keywords Bullen-type inequalities, Hermite-Hadamard-type inequalities, c-onvex functions, Riemann-Liouville fractional integrals, special means

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1. Introduction

The theory of convex functions, which is a mathematical discipline, has been widely applied in various fields including optimization theory, control theory, energy systems, information theory and physics. Furthermore, convexity theory and its applications have offered solutions to numerous problems arising from various areas of mathematics. In particular, the emergence of many basic inequalities from convex functions has led to the rapid advancement of these functions. Besides, the application of integral inequalities satisfying the idea of convexity has been broadly utilized to achieve many novel outcomes in the theory of inequalities. The first essential outcome for a convex function is known as the Hermite-Hadamard inequality, which was investigated by C. Hermite and J. Hadamard [14,21] stated as follows:

$$\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right) \leq \frac{1}{\mathfrak{h}-\mathfrak{g}} \int\limits_{\mathfrak{g}}^{\mathfrak{h}} \Xi(\pi) d\pi \leq \frac{\Xi(\mathfrak{g})+\Xi(\mathfrak{h})}{2}, \tag{1.1}$$

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where $\Xi : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\mathfrak{g}, \mathfrak{h} \in I$ with $\mathfrak{g} < \mathfrak{h}$. If Ξ is a concave function, then two inequalities in the expression apply in the opposite direction.

For the average value of a convex function across a compact interval, the Hermite-Hadamard inequality gives both upper and lower bounds. Accordingly, this inequality is utilized in many branches of mathematics such as integral calculus, probability theory, statistics, and optimization, and also acts as a foundation for numerous other inequalities. Additionally, it is used to solve real-world problems in physics, engineering, economics, and other disciplines. The applications of this inequality keep on developing as new problems arise, making it an indispensable tool for resolving complex mathematical problems as well as issues from various fields of study. On the other hand, the Hermite-Hadamard inequality is identified by the trapezoid inequality on its right part and the midpoint inequality on its left part. Dragomir and Agarwal [13] initially demonstrated trapezoid-type inequalities for the case of convex functions, while Kırmacı [35] first established midpoint-type inequalities for the case of convex functions. Numerous studies have been conducted in this field since these inequalities emerged [2, 31, 42, 43].

In [7] Bullen proved an alternative Hermite-Hadamard type inequality, known as Bullen's inequality: If $\Xi : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\mathfrak{g}, \mathfrak{h} \in I$ with $\mathfrak{g} < \mathfrak{h}$, then

$$\frac{1}{\mathfrak{h}-\mathfrak{g}} \int_{\mathfrak{g}}^{\mathfrak{h}} \Xi(\pi) d\pi \leq \frac{1}{2} \left[\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right) + \frac{\Xi(\mathfrak{g}) + \Xi(\mathfrak{h})}{2} \right]. \tag{1.2}$$

The most important objective of these kinds of inequalities is that they provide more accurate and stronger information about error estimates of the widely examined quadrature and cubature rules. The inequality (1.2) also gives the error bounds for the remainder of Bullen quadrature schemes. So, researchers have focused their studies on these types of inequalities. For instance, Çakmak [12] obtained various novel inequalities for differentiable functions based on h-convexity including Bullen-type inequalities. Dragomir and Wang [15] established a natural extension of this inequalities. In [30], İşcan et al. presented a new general identity for differentiable functions and therefore obtained certain novel inequalities, including those of the Hermite-Hadamard-type and Bullen-type. Tseng et al. [45] used Lipschitz functions to study a few Hadamard-type and Bullen-type inequalities and created several applications through the use of special means. Following the appearance of these inequalities, considerable research has been carried out in this field [37, 47].

While classical derivatives and integrals solve many problems in science and technology, they can be inadequate in certain cases. Fractional calculus dealing with derivatives and integrals of arbitrary order presents new ways to tackle these problems. Then, this approach offers a foundation for exploring and understanding systems governed by fractional dynamics, enabling a more detailed mathematical description of complex phenomena, see [8, 9, 11, 24, 32, 36, 39, 41]. Hence, with the advent of fractional integrals and derivatives such as Riemann-Liouville [20, 34], Caputo-Fabrizio [10], Atangana-Baleanu [3], tempered [40] and conformable [33], this calculus has become more prominent and has been utilized in diverse fields of science and engineering. Besides, many researchers have used fractional integral operators to establish different kinds of integral inequalities.

Throughout this paper, we will utilize the Riemann-Liouville fractional integrals that are shown below:

Definition 1.1. For $\Xi \in L_1[\mathfrak{g}, \mathfrak{h}]$, the Riemann-Liouville integrals of order $\varpi > 0$ are given by

$$J_{\mathfrak{g}+}^{\varpi}\Xi(\pi) = \frac{1}{\Gamma(\varpi)} \int_{\mathfrak{g}}^{\pi} (\pi - \varrho)^{\varpi - 1} \Xi(\varrho) d\varrho, \quad \pi > \mathfrak{g}$$

and

$$J_{\mathfrak{h}-}^{\varpi}\Xi(\pi) = \frac{1}{\Gamma(\varpi)} \int_{\pi}^{\mathfrak{h}} (\varrho - \pi)^{\varpi - 1} \Xi(\varrho) d\varrho, \quad \pi < \mathfrak{h}.$$

Here, $\Gamma(\varpi)$ is the Gamma function and $J_{\mathfrak{g}+}^{0}\Xi(\pi) = J_{\mathfrak{h}-}^{0}\Xi(\pi) = \Xi(\pi)$. Clearly, under the condition $\varpi = 1$, the Riemann-Liouville integrals will coincide with classical integrals.

Numerous significant developments related to this fractional integrals have been documented in the literature. Sarıkaya et al. [44] gave the following Hermite-Hadamard-type inequalities in terms of this fractional integrals:

Theorem 1.1. *Let $\Xi : [\mathfrak{g}, \mathfrak{h}] \to \mathbb{R}$ be a positive function with $0 \leq \mathfrak{g} < \mathfrak{h}$ and $\Xi \in L_1 [\mathfrak{g}, \mathfrak{h}]$. If Ξ is a convex function on $[\mathfrak{g}, \mathfrak{h}]$, then the following inequalities for fractional integrals hold:*

$$\Xi\left(\frac{\mathfrak{g} + \mathfrak{h}}{2}\right) \leq \frac{\Gamma(\varpi + 1)}{2 (\mathfrak{h} - \mathfrak{g})^{\varpi}} \left[J_{\mathfrak{g}+}^{\varpi}\Xi(\mathfrak{h}) + J_{\mathfrak{h}-}^{\varpi}\Xi(\mathfrak{g})\right] \leq \frac{\Xi(\mathfrak{g}) + \Xi(\mathfrak{h})}{2} \tag{1.3}$$

for $\varpi > 0$.

Note that in the above theorem, the condition $\Xi : [\mathfrak{g}, \mathfrak{h}] \to \mathbb{R}$ being a positive function with $0 \leq \mathfrak{g} < \mathfrak{h}$ can be weakened as $\Xi : [\mathfrak{g}, \mathfrak{h}] \to \mathbb{R}$ with $\mathfrak{g} < \mathfrak{h}$.

Theorem 1.2. *Let $\Xi : [\mathfrak{g}, \mathfrak{h}] \to \mathbb{R}$ be a differentiable mapping on $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g} < \mathfrak{h}$. If $|\Xi'|$ is convex on $[\mathfrak{g}, \mathfrak{h}]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{\Xi(\mathfrak{g}) + \Xi(\mathfrak{h})}{2} - \frac{\Gamma(\varpi + 1)}{2 (\mathfrak{h} - \mathfrak{g})^{\varpi}} \left[J_{\mathfrak{g}+}^{\varpi}\Xi(\mathfrak{h}) + J_{\mathfrak{h}-}^{\varpi}\Xi(\mathfrak{g})\right]\right| \\ &\leq \frac{2^{\varpi} - 1}{2^{\varpi+1}(\varpi + 1)} (\mathfrak{h} - \mathfrak{g}) \left(|\Xi'(\mathfrak{g})| + |\Xi'(\mathfrak{h})|\right) \end{aligned} \tag{1.4}$$

for $\varpi > 0$. It is easily to see that if we choose $\varpi = 1$ in Theorem 1.2, then inequality (1.4) turns into inequality as shown in (Theorem 2.2, [13]).

Later, Iqbal et al. [29] examined several inequalities regarding the fractional midpoint-type inequalities for the convex functions. Also, Agarwal et al. [1] studied many Hermite-Hadamard type inequalities with the aid of generalized k-fractional integrals. Moreover, Hwang and Tseng [28] generalized many of the previously known Hermite-Hadamard type inequalities concerning fractional integrals. For information on Hermite-Hadamard-type inequalities that involve different fractional integrals, one can consult [4, 6, 23, 38, 46].

In 2016, Hwang et al. [27] proposed a new upper bound associated with Bullen type inequality in the following manner:

Theorem 1.3. *Let $\Xi:[\mathfrak{g},\mathfrak{h}]\to\mathbb{R}$ be a differentiable mapping on $(\mathfrak{g},\mathfrak{h})$ with $\mathfrak{g}<\mathfrak{h}$. If $|\Xi'|$ is convex on $[\mathfrak{g},\mathfrak{h}]$, then we have:*

$$\left|\frac{\Gamma(\varpi+1)}{2(\mathfrak{h}-\mathfrak{g})^{\varpi}}\left[J_{\mathfrak{g}+}^{\varpi}\Xi(\mathfrak{h})+J_{\mathfrak{h}-}^{\varpi}\Xi(\mathfrak{g})\right]-\left[\frac{3^{\varpi}-1}{4^{\varpi}}\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)+\frac{4^{\varpi}-3^{\varpi}+1}{4^{\varpi}}\frac{\Xi(\mathfrak{g})+\Xi(\mathfrak{h})}{2}\right]\right|$$
$$\leq\frac{1}{\varpi+1}\left(\frac{2^{\varpi}+1}{2^{\varpi+1}}-\frac{3^{\varpi+1}+1}{4^{\varpi+1}}\right)(\mathfrak{h}-\mathfrak{g})\left(|\Xi'(\mathfrak{g})|+|\Xi'(\mathfrak{h})|\right)$$

for $\varpi>0$. It is evident that if we adopt $\varpi=1$ in Theorem 1.3, then this inequality becomes a Bullen-type inequality.

Recently, numerous significant developments related to Bullen-type inequalities based on distinct fractional integral operators have been documented in the literature. Erden and Sarikaya [17] examined generalized Bullen-type inequalities concerning local fractional integrals. Hussain and Mehboob [25] presented new estimates for the Bullen-type inequalities for (s,p)-convex functions using the generalized fractional integral identity. Then, Boulares et al. [5], with the aid of multiplicative calculus, obtained some Bullen-type inequalities, along with many applications. Hezenci et al. [22] investigated several Bullen-type inequalities for differentiable convex functions with the help of conformable fractional integrals. In paper [16], Du et al. utilized the generalized fractional integrals to explore Bullen-type inequalities. Next, Zhao et al. [48] derived the fractional forms of Bullen's inequality via a different identity. For some current results associated with these types of inequalities, see [19, 26].

The main motivation of this paper is that we will establish some new generalizations of classical Bullen's type inequalities involving the idea of the Riemann-Liouville fractional integrals for twice-differentiable mappings and some novel applications to special functions. To achieve our desired outcomes, we give a fractional integral identity for the case of differentiable mappings. This identity serves as a crucial tool in the derivation of various Bullen-type fractional integral inequalities via twice-differentiable mappings. Next, we present some important inequalities by taking into account the convexity and the Hölder inequality. Also, we provide illustrative examples accompanied by graphical representations in order to show that our main results are correct. Moreover, we apply these inequalities to special means of two positive numbers. Finally, by selecting specific choices, we analyze several well-known results in the literature.

2. Main results

Here, we start by obtaining a new extension of the Bullen-type inequalities through the usage of the Riemann-Liouville integral operators as demonstrated below:

Theorem 2.1. *Let $\Xi:[\mathfrak{g},\mathfrak{h}]\to\mathbb{R}$ be a function with $\mathfrak{g}<\mathfrak{h}$ and $\Xi\in L_1[\mathfrak{g},\mathfrak{h}]$. If Ξ is a convex function on $[\mathfrak{g},\mathfrak{h}]$, then the following Bullen's inequalities for fractional integrals hold:*

$$\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\leq\frac{1}{2}\left[\Xi\left(\frac{3\mathfrak{g}+\mathfrak{h}}{4}\right)+\Xi\left(\frac{\mathfrak{g}+3\mathfrak{h}}{4}\right)\right] \tag{2.1}$$
$$\leq\frac{\Gamma(\varpi+1)}{2^{2-\varpi}(\mathfrak{h}-\mathfrak{g})^{\varpi}}\left[J_{\mathfrak{g}+}^{\varpi}\Xi(\mathfrak{h})+J_{\mathfrak{h}-}^{\varpi}\Xi(\mathfrak{g})\right]$$

$$\leq \frac{\varpi}{2^{3-\varpi}}\left(1+\frac{1}{2^{\varpi-1}}\right)\left[\frac{\Xi(\mathfrak{g})+\Xi(\mathfrak{h})}{2}+\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right]$$
$$\leq \frac{\varpi}{2^{2-\varpi}}\left(1+\frac{1}{2^{\varpi-1}}\right)\left[\frac{\Xi(\mathfrak{g})+\Xi(\mathfrak{h})}{2}\right]$$

with $\varpi > 1$.

Proof. Since Ξ is a convex function on $[\mathfrak{g}, \frac{\mathfrak{g}+\mathfrak{h}}{2}]$, by using the first inequality in (1.3), we have

$$\Xi\left(\frac{3\mathfrak{g}+\mathfrak{h}}{4}\right) \tag{2.2}$$
$$\leq \frac{\Gamma(\varpi+1)}{2^{1-\varpi}(\mathfrak{h}-\mathfrak{g})^{\varpi}}\left[\frac{1}{\Gamma(\varpi)}\int_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}}\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}-\pi\right)^{\varpi-1}\Xi(\pi)\,d\pi+\frac{1}{\Gamma(\varpi)}\int_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}}(\pi-\mathfrak{g})^{\varpi-1}\Xi(\pi)\,d\pi\right].$$

Due to $\varpi > 1$, for all $\pi \in [\mathfrak{g}, \frac{\mathfrak{g}+\mathfrak{h}}{2}]$, we obtain $\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}-\pi\right)^{\varpi-1} \leq (\mathfrak{h}-\pi)^{\varpi-1}$. Therefore,

$$\frac{1}{\Gamma(\varpi)}\int_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}}\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}-\pi\right)^{\varpi-1}\Xi(\pi)\,d\pi \leq \frac{1}{\Gamma(\varpi)}\int_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}}(\mathfrak{h}-\pi)^{\varpi-1}\Xi(\pi)\,d\pi. \tag{2.3}$$

Substituting the inequality (2.3) into the inequality (2.2), we get

$$\Xi\left(\frac{3\mathfrak{g}+\mathfrak{h}}{4}\right) \tag{2.4}$$
$$\leq \frac{\Gamma(\varpi+1)}{2^{1-\varpi}(\mathfrak{h}-\mathfrak{g})^{\varpi}}\left[\frac{1}{\Gamma(\varpi)}\int_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}}(\mathfrak{h}-\pi)^{\varpi-1}\Xi(\pi)\,d\pi+\frac{1}{\Gamma(\varpi)}\int_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}}(\pi-\mathfrak{g})^{\varpi-1}\Xi(\pi)\,d\pi\right].$$

Similarly, by the convexity of Ξ on $[\frac{\mathfrak{g}+\mathfrak{h}}{2}, \mathfrak{h}]$, and from the first inequality in (1.3) it follows that

$$\Xi\left(\frac{\mathfrak{g}+3\mathfrak{h}}{4}\right)$$
$$\leq \frac{\Gamma(\varpi+1)}{2^{1-\varpi}(\mathfrak{h}-\mathfrak{g})^{\varpi}}\left[\frac{1}{\Gamma(\varpi)}\int_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}}(\mathfrak{h}-\pi)^{\varpi-1}\Xi(\pi)\,d\pi+\frac{1}{\Gamma(\varpi)}\int_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}}\left(\pi-\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)^{\varpi-1}\Xi(\pi)\,d\pi\right].$$

Since $\varpi > 1$, for all $\pi \in [\frac{\mathfrak{g}+\mathfrak{h}}{2}, \mathfrak{h}]$, we obtain $\left(\pi-\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)^{\varpi-1} \leq (\pi-\mathfrak{g})^{\varpi-1}$. So,

$$\frac{1}{\Gamma(\varpi)}\int_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}}\left(\pi-\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)^{\varpi-1}\Xi(\pi)\,d\pi \leq \frac{1}{\Gamma(\varpi)}\int_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}}(\pi-\mathfrak{g})^{\varpi-1}\Xi(\pi)\,d\pi.$$

Hence, we arrive at

$$\begin{aligned}
&\Xi\left(\frac{\mathfrak{g}+3\mathfrak{h}}{4}\right)\\
\leq&\frac{\Gamma(\varpi+1)}{2^{1-\varpi}\left(\mathfrak{h}-\mathfrak{g}\right)^{\varpi}}\left[\frac{1}{\Gamma(\varpi)}\int_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}}\left(\mathfrak{h}-\pi\right)^{\varpi-1}\Xi\left(\pi\right)d\pi+\frac{1}{\Gamma(\varpi)}\int_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}}\left(\pi-\mathfrak{g}\right)^{\varpi-1}\Xi\left(\pi\right)d\pi\right].
\end{aligned}\tag{2.5}$$

Together with the inequalities (2.4) and (2.5), we obtain

$$\Xi\left(\frac{\mathfrak{g}+3\mathfrak{h}}{4}\right)+\Xi\left(\frac{3\mathfrak{g}+\mathfrak{h}}{4}\right)\leq\frac{\Gamma(\varpi+1)}{2^{1-\varpi}\left(\mathfrak{h}-\mathfrak{g}\right)^{\varpi}}\left[J_{\mathfrak{g}+}^{\varpi}\Xi(\mathfrak{h})+J_{\mathfrak{h}-}^{\varpi}\Xi(\mathfrak{g})\right].\tag{2.6}$$

Because Ξ is the convex function on $[\mathfrak{g},\mathfrak{h}]$, we can derive

$$\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\leq\frac{1}{2}\left[\Xi\left(\frac{\mathfrak{g}+3\mathfrak{h}}{4}\right)+\Xi\left(\frac{3\mathfrak{g}+\mathfrak{h}}{4}\right)\right].$$

By placing the above result into (2.6), we achieve the first inequality in (2.1).

Now, it is straightforward to confirm that the following inequality holds true for all $\pi\in[\mathfrak{g},\frac{\mathfrak{g}+\mathfrak{h}}{2}]$ and $\varpi>1$:

$$\left(\mathfrak{h}-\pi\right)^{\varpi-1}+\left(\pi-\mathfrak{g}\right)^{\varpi-1}\leq\left(\mathfrak{h}-\mathfrak{g}\right)^{\varpi-1}\left(1+\frac{1}{2^{\varpi-1}}\right).$$

In the same manner, the following inequality is satisfied for all $\pi\in[\frac{\mathfrak{g}+\mathfrak{h}}{2},\mathfrak{h}]$ and $\varpi>1$:

$$\left(\mathfrak{h}-\pi\right)^{\varpi-1}+\left(\pi-\mathfrak{g}\right)^{\varpi-1}\leq\left(\mathfrak{h}-\mathfrak{g}\right)^{\varpi-1}\left(1+\frac{1}{2^{\varpi-1}}\right).$$

Due to the convexity of Ξ on $[\mathfrak{g},\frac{\mathfrak{g}+\mathfrak{h}}{2}]$ and $[\frac{\mathfrak{g}+\mathfrak{h}}{2},\mathfrak{h}]$, applying Hermite-Hadamard inequalities gives us

$$\begin{aligned}
&\frac{1}{\Gamma(\varpi)}\int_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}}\left[\left(\mathfrak{h}-\pi\right)^{\varpi-1}+\left(\pi-\mathfrak{g}\right)^{\varpi-1}\right]\Xi\left(\pi\right)d\pi \qquad (2.7)\\
\leq&\frac{\left(\mathfrak{h}-\mathfrak{g}\right)^{\varpi-1}}{\Gamma(\varpi)}\left(1+\frac{1}{2^{\varpi-1}}\right)\int_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}}\Xi\left(\pi\right)d\pi\\
\leq&\frac{\left(\mathfrak{h}-\mathfrak{g}\right)^{\varpi}}{2\Gamma(\varpi)}\left(1+\frac{1}{2^{\varpi-1}}\right)\left[\frac{\Xi\left(\mathfrak{g}\right)+\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)}{2}\right]
\end{aligned}$$

and

$$\frac{1}{\Gamma(\varpi)}\int_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}}\left[\left(\mathfrak{h}-\pi\right)^{\varpi-1}+\left(\pi-\mathfrak{g}\right)^{\varpi-1}\right]\Xi\left(\pi\right)d\pi\tag{2.8}$$

$$\leq \frac{(\mathfrak{h}-\mathfrak{g})^{\varpi-1}}{\Gamma(\varpi)}\left(1+\frac{1}{2^{\varpi-1}}\right)\int\limits_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}}\Xi(\pi)\,d\pi$$

$$\leq \frac{(\mathfrak{h}-\mathfrak{g})^{\varpi}}{2\Gamma(\varpi)}\left(1+\frac{1}{2^{\varpi-1}}\right)\left[\frac{\Xi(\mathfrak{h})+\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)}{2}\right]$$

respectively. Taking into account both the inequalities (2.7) and (2.8), we find

$$\begin{aligned}&\frac{\Gamma(\varpi+1)}{2^{2-\varpi}(\mathfrak{h}-\mathfrak{g})^{\varpi}}\left[J_{\mathfrak{g}+}^{\varpi}\Xi(\mathfrak{h})+J_{\mathfrak{h}-}^{\varpi}\Xi(\mathfrak{g})\right]\\ &\leq \frac{\varpi}{2^{3-\varpi}}\left(1+\frac{1}{2^{\varpi-1}}\right)\left[\frac{\Xi(\mathfrak{g})+\Xi(\mathfrak{h})}{2}+\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right]\\ &\leq \frac{\varpi}{2^{2-\varpi}}\left(1+\frac{1}{2^{\varpi-1}}\right)\left[\frac{\Xi(\mathfrak{g})+\Xi(\mathfrak{h})}{2}\right],\end{aligned}$$

which implies the desired second inequality in (2.1). □

Remark 2.1. For $\varpi \to 1$ in Theorem 2.1, the inequality (2.1) transforms into the inequality (1.2).

Now, we employ the following lemma to develop several integral inequalities linked to Bullen-type inequalities through Riemann-Liouville integral operators. So, this lemma is crucial.

Lemma 2.1. *Let $\Xi : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I^o, the interior of the interval I, where $\mathfrak{g},\mathfrak{h} \in I^o$ satisfying $\mathfrak{g} < \mathfrak{h}$ and let $\Xi, \Xi', \Xi'' \in L_1[\mathfrak{g},\mathfrak{h}]$. Then, for $\varpi > 1$ and $\pi \in (\mathfrak{g},\mathfrak{h})$, the following identity is satisfied:*

$$\begin{aligned}&\frac{\Xi(\pi)}{2(\mathfrak{h}-\mathfrak{g})^{1-\varpi}}+\frac{\varpi}{4(\mathfrak{h}-\mathfrak{g})^{2-\varpi}}\left[(\pi-\mathfrak{g})\Xi(\mathfrak{g})+(\mathfrak{h}-\pi)\Xi(\mathfrak{h})\right]\\ &-\frac{\Gamma(\varpi+2)}{4(\mathfrak{h}-\mathfrak{g})}\left[J_{\mathfrak{g}^+}^{\varpi}\Xi(\mathfrak{h})+J_{\mathfrak{h}^-}^{\varpi}\Xi(\mathfrak{g})\right]\\ &+\frac{\Gamma(\varpi+1)}{4(\mathfrak{h}-\mathfrak{g})}\left[(\mathfrak{h}-\pi)J_{\mathfrak{g}^+}^{\varpi-1}\Xi(\mathfrak{h})+(\pi-\mathfrak{g})J_{\mathfrak{h}^-}^{\varpi-1}\Xi(\mathfrak{g})\right]\\ =&-\frac{1}{4(\mathfrak{h}-\mathfrak{g})}\sum_{k=1}^{4}C_k(\varpi,\pi),\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}C_1(\varpi,\pi)&=\int\limits_{\mathfrak{g}}^{\pi}(\varrho-\mathfrak{g})^{\varpi}(\varrho-\pi)\Xi''(\varrho)d\varrho,\\ C_2(\varpi,\pi)&=-\int\limits_{\pi}^{\mathfrak{h}}(\mathfrak{h}-\varrho)^{\varpi}(\varrho-\pi)\Xi''(\varrho)d\varrho,\\ C_3(\varpi,\pi)&=\int\limits_{\pi}^{\mathfrak{h}}(\varrho-\pi)\left((\varrho-\mathfrak{g})^{\varpi}-(\mathfrak{h}-\mathfrak{g})^{\varpi}\right)\Xi''(\varrho)d\varrho,\\ C_4(\varpi,\pi)&=\int\limits_{\mathfrak{g}}^{\pi}(\varrho-\pi)\left((\mathfrak{h}-\mathfrak{g})^{\varpi}-(\mathfrak{h}-\varrho)^{\varpi}\right)\Xi''(\varrho)d\varrho.\end{aligned}$$

Proof. By integration by parts, we have

$C_1(\varpi,\pi)$

$$= \int_{\mathfrak{g}}^{\pi} (\varrho - \mathfrak{g})^{\varpi} (\varrho - \pi) \Xi''(\varrho) d\varrho$$

$$= -(\pi - \mathfrak{g})^{\varpi} \Xi(\pi) + \int_{\mathfrak{g}}^{\pi} \left[2\varpi(\varrho - \mathfrak{g})^{\varpi-1} + \varpi(\varpi - 1)(\varrho - \pi)(\varrho - \mathfrak{g})^{\varpi-2}\right] \Xi(\varrho) d\varrho$$

$$= -(\pi - \mathfrak{g})^{\varpi} \Xi(\pi) + \int_{\mathfrak{g}}^{\pi} \left[\varpi(\varpi + 1)(\varrho - \mathfrak{g})^{\varpi-1} - \varpi(\varpi - 1)(\pi - \mathfrak{g})(\varrho - \mathfrak{g})^{\varpi-2}\right] \Xi(\varrho) d\varrho$$

and

$$C_2(\varpi, \pi)$$

$$= -\int_{\pi}^{\mathfrak{h}} (\mathfrak{h} - \varrho)^{\varpi} (\varrho - \pi) \Xi''(\varrho) d\varrho$$

$$= -(\mathfrak{h} - \pi)^{\varpi} \Xi(\pi) + \int_{\pi}^{\mathfrak{h}} \left[2\varpi(\mathfrak{h} - \varrho)^{\varpi-1} - \varpi(\varpi - 1)(\varrho - \pi)(\mathfrak{h} - \varrho)^{\varpi-2}\right] \Xi(\varrho) d\varrho$$

$$= -(\mathfrak{h} - \pi)^{\varpi} \Xi(\pi) + \int_{\pi}^{\mathfrak{h}} \left[\varpi(\varpi+1)(\mathfrak{h} - \varrho)^{\varpi-1} - \varpi(\varpi - 1)(\mathfrak{h} - \pi)(\mathfrak{h} - \varrho)^{\varpi-2}\right] \Xi(\varrho) d\varrho.$$

Similarly, we obtain

$$\begin{aligned} C_3(\varpi, \pi) &= \int_{\pi}^{\mathfrak{h}} (\varrho - \pi) \left((\varrho - \mathfrak{g})^{\varpi} - (\mathfrak{h} - \mathfrak{g})^{\varpi}\right) \Xi''(\varrho) d\varrho \\ &= -\varpi(\mathfrak{h} - \pi)(\mathfrak{h} - \mathfrak{g})^{\varpi-1} \Xi(\mathfrak{h}) + \left((\pi - \mathfrak{g})^{\varpi} - (\mathfrak{h} - \mathfrak{g})^{\varpi}\right) \Xi(\pi) \\ &\quad + \int_{\pi}^{\mathfrak{h}} \left[\varpi(\varpi + 1)(\varrho - \mathfrak{g})^{\varpi-1} - \varpi(\varpi - 1)(\pi - \mathfrak{g})(\varrho - \mathfrak{g})^{\varpi-2}\right] \Xi(\varrho) d\varrho \end{aligned}$$

and

$$\begin{aligned} C_4(\varpi, \pi) &= \int_{\mathfrak{g}}^{\pi} (\varrho - \pi) \left((\mathfrak{h} - \mathfrak{g})^{\varpi} - (\mathfrak{h} - \varrho)^{\varpi}\right) \Xi''(\varrho) d\varrho \\ &= -\varpi(\pi - \mathfrak{g})(\mathfrak{h} - \mathfrak{g})^{\varpi-1} \Xi(\mathfrak{g}) - \left((\mathfrak{h} - \mathfrak{g})^{\varpi} - (\mathfrak{h} - \pi)^{\varpi}\right) \Xi(\pi) \\ &\quad + \int_{\mathfrak{g}}^{\pi} \left[\varpi(\varpi + 1)(\mathfrak{h} - \varrho)^{\varpi-1} - \varpi(\varpi - 1)(\mathfrak{h} - \pi)(\mathfrak{h} - \varrho)^{\varpi-2}\right] \Xi(\varrho) d\varrho. \end{aligned}$$

The required equality (2.9) can be achieved by summing the equalities above and multiplying the result by $-\frac{1}{4(\mathfrak{h}-\mathfrak{g})}$. □

Corollary 2.1. *If we choose* $\pi = \frac{\mathfrak{g}+\mathfrak{h}}{2}$ *in Lemma 2.1, then the equality (2.9) converts to the following equality*

$$-\frac{1}{4(\mathfrak{h} - \mathfrak{g})} \left(C_1\left(\varpi, \frac{\mathfrak{g} + \mathfrak{h}}{2}\right) + C_2\left(\varpi, \frac{\mathfrak{g} + \mathfrak{h}}{2}\right) + C_3\left(\varpi, \frac{\mathfrak{g} + \mathfrak{h}}{2}\right) + C_4\left(\varpi, \frac{\mathfrak{g} + \mathfrak{h}}{2}\right)\right)$$

$$= \frac{1}{4(\mathfrak{h}-\mathfrak{g})^{1-\varpi}}\left[2\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)+\varpi\left(\frac{\Xi(\mathfrak{g})+\Xi(\mathfrak{h})}{2}\right)\right]$$
$$-\frac{\Gamma(\varpi+2)}{4(\mathfrak{h}-\mathfrak{g})}\left[J_{\mathfrak{g}^+}^{\varpi}\Xi(\mathfrak{h})+J_{\mathfrak{h}^-}^{\varpi}\Xi(\mathfrak{g})\right]+\frac{\Gamma(\varpi+1)}{8}\left[J_{\mathfrak{g}^+}^{\varpi-1}\Xi(\mathfrak{h})+J_{\mathfrak{h}^-}^{\varpi-1}\Xi(\mathfrak{g})\right].$$

Corollary 2.2. *In Lemma 2.1, if we select $\pi = \frac{\mathfrak{g}+\mathfrak{h}}{2}$ as ϖ tends to 1, then the equality (2.9) can be expressed as follows:*

$$-\frac{1}{4(\mathfrak{h}-\mathfrak{g})}\left(C_1\left(1,\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)+C_2\left(1,\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)+C_3\left(1,\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)+C_4\left(1,\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right)$$
$$=\frac{1}{2}\left[\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)+\frac{\Xi(\mathfrak{g})+\Xi(\mathfrak{h})}{2}\right]-\frac{1}{\mathfrak{h}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{h}}\Xi(\varrho)d\varrho.$$

Theorem 2.2. *Assume that all the conditions of Lemma 2.1 hold. If $|\Xi''|$ is a convex function on $[\mathfrak{g},\mathfrak{h}]$, then the following inequality is satisfied:*

$$\left|\frac{(\mathfrak{h}-\mathfrak{g})^{\varpi-1}}{2}\Xi(\pi)+\frac{\varpi(\mathfrak{h}-\mathfrak{g})^{\varpi-2}}{4}\left[(\pi-\mathfrak{g})\Xi(\mathfrak{g})+(\mathfrak{h}-\pi)\Xi(\mathfrak{h})\right]\right. \tag{2.10}$$
$$\left.-\frac{\Gamma(\varpi+2)}{4(\mathfrak{h}-\mathfrak{g})}\left[J_{\mathfrak{g}^+}^{\varpi}\Xi(\mathfrak{h})+J_{\mathfrak{h}^-}^{\varpi}\Xi(\mathfrak{g})\right]+\frac{\Gamma(\varpi+1)}{4(\mathfrak{h}-\mathfrak{g})}\left[(\mathfrak{h}-\pi)J_{\mathfrak{g}^+}^{\varpi-1}\Xi(\mathfrak{h})+(\pi-\mathfrak{g})J_{\mathfrak{h}^-}^{\varpi-1}\Xi(\mathfrak{g})\right]\right|$$
$$\leq\frac{1}{4(\mathfrak{h}-\mathfrak{g})}\left\{\left[\frac{(\pi-\mathfrak{g})^{\varpi+3}(\mathfrak{h}-\mathfrak{g})+(\mathfrak{h}-\pi)^{\varpi+3}(\mathfrak{h}-\mathfrak{g})-(\mathfrak{h}-\mathfrak{g})^{\varpi+4}}{(\varpi+2)(\varpi+3)(\pi-\mathfrak{g})(\mathfrak{h}-\pi)}\right.\right.$$
$$+\frac{(\mathfrak{h}-\mathfrak{g})^{\varpi}(\mathfrak{h}-\pi)^2+(\mathfrak{h}-\mathfrak{g})^{\varpi}(\pi-\mathfrak{g})^2}{6}$$
$$\left.+\frac{(\pi-\mathfrak{g})^2(\mathfrak{h}-\mathfrak{g})^{\varpi+2}-(\pi-\mathfrak{g})^{\varpi+3}(\mathfrak{h}-\mathfrak{g})+(\mathfrak{h}-\pi)^2(\mathfrak{h}-\mathfrak{g})^{\varpi+2}-(\mathfrak{h}-\mathfrak{g})(\mathfrak{h}-\pi)^{\varpi+3}}{(\varpi+1)(\varpi+2)(\mathfrak{h}-\pi)(\pi-\mathfrak{g})}\right]$$
$$\times\left|\Xi''(\pi)\right|+\left[\frac{(\mathfrak{h}-\pi)^{\varpi+3}-(\mathfrak{h}-\pi)^2(\mathfrak{h}-\mathfrak{g})^{\varpi+1}}{(\pi-\mathfrak{g})(\varpi+1)}+\frac{2(\mathfrak{h}-\pi)(\mathfrak{h}-\mathfrak{g})^{\varpi+2}-2(\mathfrak{h}-\pi)^{\varpi+3}}{(\pi-\mathfrak{g})(\varpi+2)}\right.$$
$$\left.+\frac{(\mathfrak{h}-\pi)^{\varpi+3}-(\mathfrak{h}-\mathfrak{g})^{\varpi+3}}{(\pi-\mathfrak{g})(\varpi+3)}+\frac{2(\pi-\mathfrak{g})^{\varpi+2}}{(\varpi+1)(\varpi+2)(\varpi+3)}+\frac{(\pi-\mathfrak{g})^2(\mathfrak{h}-\mathfrak{g})^{\varpi}}{3}\right]\left|\Xi''(\mathfrak{g})\right|$$
$$+\left[\frac{(\pi-\mathfrak{g})^{\varpi+3}-(\pi-\mathfrak{g})^2(\mathfrak{h}-\mathfrak{g})^{\varpi+1}}{(\mathfrak{h}-\pi)(\varpi+1)}+\frac{2(\pi-\mathfrak{g})(\mathfrak{h}-\mathfrak{g})^{\varpi+2}-2(\pi-\mathfrak{g})^{\varpi+3}}{(\mathfrak{h}-\pi)(\varpi+2)}\right.$$
$$\left.\left.+\frac{(\pi-\mathfrak{g})^{\varpi+3}-(\mathfrak{h}-\mathfrak{g})^{\varpi+3}}{(\mathfrak{h}-\pi)(\varpi+3)}+\frac{2(\mathfrak{h}-\pi)^{\varpi+2}}{(\varpi+1)(\varpi+2)(\varpi+3)}+\frac{(\mathfrak{h}-\pi)^2(\mathfrak{h}-\mathfrak{g})^{\varpi}}{3}\right]\left|\Xi''(\mathfrak{h})\right|\right\},$$

where $\varpi > 1$ and $\pi \in (\mathfrak{g},\mathfrak{h})$.

Proof. Given that the function $|\Xi''|$ is convex, the following inequalities are valid:

$$|\Xi''(\varrho)|\leq\frac{\varrho-\mathfrak{g}}{\pi-\mathfrak{g}}\left|\Xi''(\pi)\right|+\frac{\pi-\varrho}{\pi-\mathfrak{g}}\left|\Xi''(\mathfrak{g})\right|, \tag{2.11}$$

$$|\Xi''(\varrho)|\leq\frac{\mathfrak{h}-\varrho}{\mathfrak{h}-\pi}\left|\Xi''(\pi)\right|+\frac{\varrho-\pi}{\mathfrak{h}-\pi}\left|\Xi''(\mathfrak{h})\right|. \tag{2.12}$$

With the Lemma 2.1 and the convexity of the function $|\Xi''|$, we deduce

$$\left|\frac{(\mathfrak{h}-\mathfrak{g})^{\varpi-1}}{2}\Xi(\pi)+\frac{\varpi(\mathfrak{h}-\mathfrak{g})^{\varpi-2}}{4}\left[(\pi-\mathfrak{g})\Xi(\mathfrak{g})+(\mathfrak{h}-\pi)\Xi(\mathfrak{h})\right]\right. \tag{2.13}$$

$$
\begin{aligned}
&-\frac{\Gamma(\varpi+2)}{4(\mathfrak{h}-\mathfrak{g})}\left[J_{\mathfrak{g}+}^{\varpi}\Xi(\mathfrak{h})+J_{\mathfrak{h}-}^{\varpi}\Xi(\mathfrak{g})\right] \\
&+\frac{\Gamma(\varpi+1)}{4(\mathfrak{h}-\mathfrak{g})}\left[(\mathfrak{h}-\pi)J_{\mathfrak{g}+}^{\varpi-1}\Xi(\mathfrak{h})+(\pi-\mathfrak{g})J_{\mathfrak{h}-}^{\varpi-1}\Xi(\mathfrak{g})\right]\Bigg| \\
\leq\ & \frac{1}{4(\mathfrak{h}-\mathfrak{g})}\left[\int_{\mathfrak{g}}^{\pi}(\varrho-\mathfrak{g})^{\varpi}(\pi-\varrho)\left|\Xi''(\varrho)\right|d\varrho+\int_{\pi}^{\mathfrak{h}}(\mathfrak{h}-\varrho)^{\varpi}(\varrho-\pi)\left|\Xi''(\varrho)\right|d\varrho\right. \\
&+\int_{\mathfrak{g}}^{\pi}(\pi-\varrho)\left[(\mathfrak{h}-\mathfrak{g})^{\varpi}-(\mathfrak{h}-\varrho)^{\varpi}\right]\left|\Xi''(\varrho)\right|d\varrho \\
&\left.+\int_{\pi}^{\mathfrak{h}}(\varrho-\pi)\left[(\mathfrak{h}-\mathfrak{g})^{\varpi}-(\varrho-\mathfrak{g})^{\varpi}\right]\left|\Xi''(\varrho)\right|d\varrho\right] \\
\leq\ & \frac{1}{4(\mathfrak{h}-\mathfrak{g})}\left[\frac{|\Xi''(\pi)|}{\pi-\mathfrak{g}}\int_{\mathfrak{g}}^{\pi}(\varrho-\mathfrak{g})^{\varpi+1}(\pi-\varrho)d\varrho+\frac{|\Xi''(\mathfrak{g})|}{\pi-\mathfrak{g}}\int_{\mathfrak{g}}^{\pi}(\pi-\varrho)^{2}(\varrho-\mathfrak{g})^{\varpi}d\varrho\right. \\
&+\frac{|\Xi''(\pi)|}{\mathfrak{h}-\pi}\int_{\pi}^{\mathfrak{h}}(\mathfrak{h}-\varrho)^{\varpi+1}(\varrho-\pi)d\varrho+\frac{|\Xi''(\mathfrak{h})|}{\mathfrak{h}-\pi}\int_{\pi}^{\mathfrak{h}}(\pi-\varrho)^{2}(\mathfrak{h}-\varrho)^{\varpi}d\varrho \\
&+\frac{|\Xi''(\pi)|}{\pi-\mathfrak{g}}\int_{\mathfrak{g}}^{\pi}(\pi-\varrho)(\varrho-\mathfrak{g})\left[(\mathfrak{h}-\mathfrak{g})^{\varpi}-(\mathfrak{h}-\varrho)^{\varpi}\right]d\varrho \\
&+\frac{|\Xi''(\mathfrak{g})|}{\pi-\mathfrak{g}}\int_{\mathfrak{g}}^{\pi}(\pi-\varrho)^{2}\left[(\mathfrak{h}-\mathfrak{g})^{\varpi}-(\mathfrak{h}-\varrho)^{\varpi}\right]d\varrho \\
&+\frac{|\Xi''(\pi)|}{\mathfrak{h}-\pi}\int_{\pi}^{\mathfrak{h}}(\varrho-\pi)(\mathfrak{h}-\varrho)\left[(\mathfrak{h}-\mathfrak{g})^{\varpi}-(\varrho-\mathfrak{g})^{\varpi}\right]d\varrho \\
&\left.+\frac{|\Xi''(\mathfrak{h})|}{\mathfrak{h}-\pi}\int_{\pi}^{\mathfrak{h}}(\pi-\varrho)^{2}\left[(\mathfrak{h}-\mathfrak{g})^{\varpi}-(\varrho-\mathfrak{g})^{\varpi}\right]d\varrho\right].
\end{aligned}
$$

Below is the method for calculating the integral $\int_{\mathfrak{g}}^{\pi}(\varrho-\mathfrak{g})^{\varpi+1}(\pi-\varrho)d\varrho$, which is one of the integrals in inequality (2.13):

$$
\begin{aligned}
\int_{\mathfrak{g}}^{\pi}(\varrho-\mathfrak{g})^{\varpi+1}(\pi-\varrho)d\varrho &= \int_{\mathfrak{g}}^{\pi}(\varrho-\mathfrak{g})^{\varpi+1}(\pi-\mathfrak{g})d\varrho-\int_{\mathfrak{g}}^{\pi}(\varrho-\mathfrak{g})^{\varpi+2}d\varrho \\
&= \frac{(\pi-\mathfrak{g})^{\varpi+3}}{(\varpi+2)(\varpi+3)}.
\end{aligned}
$$

The same method is used to calculate the remaining integrals in inequality (2.13). Replacing all of the calculated integrals into the inequality (2.13), the desired outcome is obtained. □

Remark 2.2. By setting $\pi = \frac{\mathfrak{g}+\mathfrak{h}}{2}$ in Theorem 2.2 as ϖ approaches 1, this occurs:

$$\begin{aligned}
&\left|\frac{1}{2}\left[\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)+\frac{\Xi(\mathfrak{g})+\Xi(\mathfrak{h})}{2}\right]-\frac{1}{\mathfrak{h}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{h}}\Xi(\varrho)d\varrho\right| \\
&\leq \frac{(\mathfrak{h}-\mathfrak{g})^2}{96}\left(\left|\Xi''\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right|+\frac{|\Xi''(\mathfrak{g})|+|\Xi''(\mathfrak{h})|}{2}\right) \\
&\leq \frac{(\mathfrak{h}-\mathfrak{g})^2}{48}\left(\frac{|\Xi''(\mathfrak{g})|+|\Xi''(\mathfrak{h})|}{2}\right),
\end{aligned}$$

which was given by Sarıkaya and Aktan in (Proposition 4, [43]), by using a different identity.

Remark 2.3. By imposing the condition $\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right) = \frac{\Xi(\mathfrak{g})+\Xi(\mathfrak{h})}{2}$ in Remark 2.2, we derive new upper bounds for midpoint inequality and trapezoid inequality, respectively,

$$\left|\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)-\frac{1}{\mathfrak{h}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{h}}\Xi(\varrho)d\varrho\right| \leq \frac{(\mathfrak{h}-\mathfrak{g})^2}{48}\left[\frac{|\Xi''(\mathfrak{g})|+|\Xi''(\mathfrak{h})|}{2}\right]$$

and

$$\left|\frac{\Xi(\mathfrak{g})+\Xi(\mathfrak{h})}{2}-\frac{1}{\mathfrak{h}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{h}}\Xi(\varrho)d\varrho\right| \leq \frac{(\mathfrak{h}-\mathfrak{g})^2}{48}\left[\frac{|\Xi''(\mathfrak{g})|+|\Xi''(\mathfrak{h})|}{2}\right],$$

which are smaller than those proven in (Proposition 1-2, [43]) under the specified condition.

To demonstrate the validity of our theorem, we provide a concrete example.

Example 2.1. Considering the function $\Xi(\varrho) = \varrho^2$ over $[0, 2]$, we can compute the right-hand side of the inequality (2.10) as follows:

$$\begin{aligned}
&\frac{2\pi^{\varpi+3}+2(2-\pi)^{\varpi+3}-2^{\varpi+4}}{4\pi(2-\pi)(\varpi+2)(\varpi+3)}+\frac{\left[2^{\varpi}+2^{\varpi-1}\right]\left[\pi^2+(2-\pi)^2\right]}{12} \\
&+\frac{\pi^2 2^{\varpi+2}-2\pi^{\varpi+3}+2^{\varpi+2}(2-\pi)^2-2(2-\pi)^{\varpi+3}}{4\pi(2-\pi)(\varpi+1)(\varpi+2)}+\frac{\pi^{\varpi+2}+(2-\pi)^{\varpi+2}}{2(\varpi+1)(\varpi+2)(\varpi+3)} \\
&+\frac{(2-\pi)^{\varpi+4}+\pi^{\varpi+4}-2^{\varpi+4}}{4\pi(2-\pi)(\varpi+3)}+\frac{2^{\varpi+3}[(2-\pi)^2+\pi^2]-2[(2-\pi)^{\varpi+4}+\pi^{\varpi+4}]}{4\pi(2-\pi)(\varpi+2)} \\
&+\frac{(2-\pi)^{\varpi+4}+\pi^{\varpi+4}-2^{\varpi+1}[(2-\pi)^3+\pi^3]}{4\pi(2-\pi)(\varpi+1)} := \Psi_1.
\end{aligned}$$

Moreover, it is clear that

$$\left|\frac{(\mathfrak{h}-\mathfrak{g})^{\varpi-1}}{2}\Xi(\pi)+\frac{\varpi(\mathfrak{h}-\mathfrak{g})^{\varpi-2}}{4}\left[(\pi-\mathfrak{g})\Xi(\mathfrak{g})+(\mathfrak{h}-\pi)\Xi(\mathfrak{h})\right]\right.$$

$$-\frac{\Gamma(\varpi+2)}{4(\mathfrak{h}-\mathfrak{g})}\left[J^{\varpi}_{\mathfrak{g}+}\Xi(\mathfrak{h})+J^{\varpi}_{\mathfrak{h}-}\Xi(\mathfrak{g})\right]$$
$$+\frac{\Gamma(\varpi+1)}{4(\mathfrak{h}-\mathfrak{g})}\left[(\mathfrak{h}-\pi)J^{\varpi-1}_{\mathfrak{g}+}\Xi(\mathfrak{h})+(\pi-\mathfrak{g})J^{\varpi-1}_{\mathfrak{h}-}\Xi(\mathfrak{g})\right]\Bigg|$$
$$=2^{\varpi-2}\pi^2+\varpi 2^{\varpi-2}(2-\pi)+\frac{\varpi(\varpi+1)}{8}\left[\frac{2^{\varpi+2}}{\varpi}-\frac{2^{\varpi+3}}{(\varpi+1)(\varpi+2)}\right]$$
$$+\frac{\varpi(\varpi-1)}{8}\left[(2-\pi)\left(\frac{2^{\varpi+1}}{\varpi-1}-\frac{2^{\varpi+2}}{\varpi}+\frac{2^{\varpi+1}}{\varpi+1}\right)+\pi\left(\frac{2^{\varpi+1}}{\varpi+1}\right)\right]:=\Psi_2.$$

Therefore, it is apparent from Figure 1 that for every π in the interval $(0,2)$ and $\varpi>1$, in the inequality (2.10), the left side is permanently less than the right side.

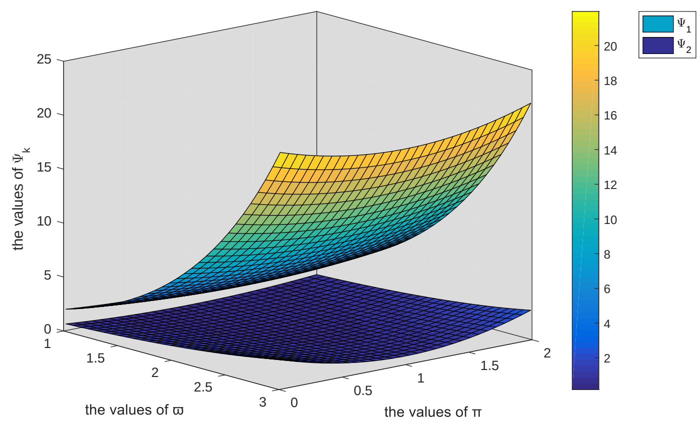

Figure 1. The graph of both sides of inequality (2.10) based on Example 2.1, calculated and plotted using MATLAB, is shown for $\varpi>1$ and $\pi\in(0,2)$.

Theorem 2.3. *Assume that all the conditions of Lemma 2.1 hold. If $|\Xi''|^q$ is a convex function on $[\mathfrak{g},\mathfrak{h}]$ for $q>1$, then the following inequality holds:*

$$\left|\frac{(\mathfrak{h}-\mathfrak{g})^{\varpi-1}}{2}\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)+\frac{\varpi(\mathfrak{h}-\mathfrak{g})^{\varpi-1}}{4}\left(\frac{\Xi(\mathfrak{g})+\Xi(\mathfrak{h})}{2}\right)\right. \tag{2.14}$$
$$\left.\frac{\Gamma(\varpi+2)}{4(\mathfrak{h}-\mathfrak{g})}\left[J^{\varpi}_{\mathfrak{g}+}\Xi(\mathfrak{h})+J^{\varpi}_{\mathfrak{h}-}\Xi(\mathfrak{g})\right]+\frac{\Gamma(\varpi+1)}{8}\left[J^{\varpi-1}_{\mathfrak{g}+}\Xi(\mathfrak{h})+J^{\varpi-1}_{\mathfrak{h}-}\Xi(\mathfrak{g})\right]\right|$$
$$\leq\left[\mathfrak{g}_{\varpi}(p,q)+\mathfrak{h}_{\varpi}(p,q)\right]$$
$$\times(\mathfrak{h}-\mathfrak{g})^{\varpi+1}\left(|\Xi''(\mathfrak{g})|+2\left|\Xi''\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right|+|\Xi''(\mathfrak{h})|\right),$$

where $\varpi>1$, $\frac{1}{p}+\frac{1}{q}=1$ and

$$\mathfrak{g}_{\varpi}(p,q)=\frac{p^{\frac{1}{p}}}{2^{\varpi+3+\frac{1}{q}}(\varpi p+1)^{\frac{1}{p}}\left[(\varpi+1)p+1\right]^{\frac{1}{p}}},$$

$$\mathfrak{h}_{\varpi}(p,q)=\frac{1}{2^{\frac{2}{q}+1}}\left(\frac{2^{p+1}-1}{2^{p+1}(p+1)}-\frac{1}{2^{p+1}}\right.$$
$$\left.-\frac{p}{2^{(\varpi+1)p+1}\left[(\varpi+1)p+1\right](\varpi p+1)}-\frac{1}{(\varpi+1)p+1}+\frac{1}{2^{p}(\varpi p+1)}\right)^{\frac{1}{p}}.$$

Proof. Applying Hölder's inequality with $\pi=\frac{\mathfrak{g}+\mathfrak{h}}{2}$ from Lemma 2.1 and the convexity property of $|\Xi''|^{q}$ with $\pi=\frac{\mathfrak{g}+\mathfrak{h}}{2}$ in the inequalities (2.11) and (2.12), we obtain

$$\left|\frac{(\mathfrak{h}-\mathfrak{g})^{\varpi-1}}{2}\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)+\frac{\varpi(\mathfrak{h}-\mathfrak{g})^{\varpi-1}}{4}\left(\frac{\Xi(\mathfrak{g})+\Xi(\mathfrak{h})}{2}\right)\right. \tag{2.15}$$
$$\left.\frac{\Gamma(\varpi+2)}{4(\mathfrak{h}-\mathfrak{g})}\left[J_{\mathfrak{g}+}^{\varpi}\Xi(\mathfrak{h})+J_{\mathfrak{h}-}^{\varpi}\Xi(\mathfrak{g})\right]+\frac{\Gamma(\varpi+1)}{8}\left[J_{\mathfrak{g}+}^{\varpi-1}\Xi(\mathfrak{h})+J_{\mathfrak{h}-}^{\varpi-1}\Xi(\mathfrak{g})\right]\right|$$
$$\leq\frac{1}{4(\mathfrak{h}-\mathfrak{g})}\left[\left(\int_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}}\left[(\varrho-\mathfrak{g})^{\varpi p}\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}-\varrho\right)^{p}\right]d\varrho\right)^{\frac{1}{p}}\right.$$
$$\times\left(\int_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}}\left[\frac{2(\varrho-\mathfrak{g})}{\mathfrak{h}-\mathfrak{g}}\left|\Xi''\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right|^{q}+\frac{2\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}-\varrho\right)}{\mathfrak{h}-\mathfrak{g}}\left|\Xi''(\mathfrak{g})\right|^{q}\right]d\varrho\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}}\left[(\mathfrak{h}-\varrho)^{\varpi p}\left(\varrho-\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)^{p}\right]d\varrho\right)^{\frac{1}{p}}$$
$$\times\left(\int_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}}\left[\frac{2(\mathfrak{h}-\varrho)}{\mathfrak{h}-\mathfrak{g}}\left|\Xi''\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right|^{q}+\frac{2\left(\varrho-\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)}{\mathfrak{h}-\mathfrak{g}}\left|\Xi''(\mathfrak{h})\right|^{q}\right]d\varrho\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}}\left[\left(\varrho-\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)^{p}\left[(\mathfrak{h}-\mathfrak{g})^{\varpi}-(\varrho-\mathfrak{g})^{\varpi}\right]^{p}\right]d\varrho\right)^{\frac{1}{p}}$$
$$\times\left(\int_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}}\left[\frac{2(\mathfrak{h}-\varrho)}{\mathfrak{h}-\mathfrak{g}}\left|\Xi''\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right|^{q}+\frac{2\left(\varrho-\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)}{\mathfrak{h}-\mathfrak{g}}\left|\Xi''(\mathfrak{h})\right|^{q}\right]d\varrho\right)^{\frac{1}{q}}$$
$$+\left(\int_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}}\left[\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}-\varrho\right)^{p}\left[(\mathfrak{h}-\mathfrak{g})^{\varpi}-(\mathfrak{h}-\varrho)^{\varpi}\right]^{p}\right]d\varrho\right)^{\frac{1}{p}}$$
$$\left.\times\left(\int_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}}\left[\frac{2(\varrho-\mathfrak{g})}{\mathfrak{h}-\mathfrak{g}}\left|\Xi''\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right|^{q}+\frac{2\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}-\varrho\right)}{\mathfrak{h}-\mathfrak{g}}\left|\Xi''(\mathfrak{g})\right|^{q}\right]d\varrho\right)^{\frac{1}{q}}\right].$$

Through a simple computation, the outcomes of certain integrals mentioned above are determined as follows:

$$\int\limits_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}}\left[\frac{2(\varrho-\mathfrak{g})}{\mathfrak{h}-\mathfrak{g}}\left|\Xi''\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right|^{q}+\frac{2\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}-\varrho\right)}{\mathfrak{h}-\mathfrak{g}}\left|\Xi''\left(\mathfrak{g}\right)\right|^{q}\right]d\varrho$$
$$=\frac{\mathfrak{h}-\mathfrak{g}}{4}\left[\left|\Xi''\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right|^{q}+\left|\Xi''\left(\mathfrak{g}\right)\right|^{q}\right],$$

$$\int\limits_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}}\left[\frac{2(\mathfrak{h}-\varrho)}{\mathfrak{h}-\mathfrak{g}}\left|\Xi''\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right|^{q}+\frac{2\left(\varrho-\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)}{\mathfrak{h}-\mathfrak{g}}\left|\Xi''\left(\mathfrak{h}\right)\right|^{q}\right]d\varrho$$
$$=\frac{\mathfrak{h}-\mathfrak{g}}{4}\left[\left|\Xi''\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right|^{q}+\left|\Xi''\left(\mathfrak{h}\right)\right|^{q}\right].$$

Also, by utilizing the property that is $\sum\limits_{k=1}^{n}(\mathfrak{g}_k+\mathfrak{h}_k)^s\leq\sum\limits_{k=1}^{n}\mathfrak{g}_k^s+\sum\limits_{k=1}^{n}\mathfrak{h}_k^s$ for $0<s\leq 1$ and $\mathfrak{g}_k,\mathfrak{h}_k\geq 0$ with $k\in\{1,2,...,n\}$, we get

$$\left[\left|\Xi''\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right|^{q}+\left|\Xi''(\mathfrak{g})\right|^{q}\right]^{\frac{1}{q}}\leq\left|\Xi''\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right|+\left|\Xi''(\mathfrak{g})\right|,$$
$$\left[\left|\Xi''\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right|^{q}+\left|\Xi''(\mathfrak{h})\right|^{q}\right]^{\frac{1}{q}}\leq\left|\Xi''\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right|+\left|\Xi''(\mathfrak{h})\right|.$$

Furthermore, it is known that we have the property

$$(\mathfrak{g}-\mathfrak{h})^p\leq\mathfrak{g}^p-\mathfrak{h}^p$$

for $\mathfrak{g}>\mathfrak{h}\geq 0$ and $p\geq 1$. This property is used to determine the remaining integrals of inequality (2.15) in the following manner:

$$\begin{aligned}\int\limits_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}}\left[(\varrho-\mathfrak{g})^{\varpi p}\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}-\varrho\right)^{p}\right]d\varrho&=\int\limits_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}}\left[(\varrho-\mathfrak{g})^{\varpi p}\left(\frac{\mathfrak{h}-\mathfrak{g}}{2}-(\varrho-\mathfrak{g})\right)^{p}\right]d\varrho\\&\leq\int\limits_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}}\left[(\varrho-\mathfrak{g})^{\varpi p}\left(\frac{\mathfrak{h}-\mathfrak{g}}{2}\right)^{p}d\varrho\right.\\&\quad\left.-\int\limits_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}}(\varrho-\mathfrak{g})^{\varpi p}(\varrho-\mathfrak{g})^{p}d\varrho\right]\\&=\left(\frac{\mathfrak{h}-\mathfrak{g}}{2}\right)^{(\varpi+1)p+1}\left(\frac{p}{(\varpi p+1)\left[(\varpi+1)p+1\right]}\right),\end{aligned}$$

$$\int\limits_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}}\left[(\mathfrak{h}-\varrho)^{\varpi p}\left(\varrho-\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)^{p}\right]d\varrho=\int\limits_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}}\left[(\mathfrak{h}-\varrho)^{\varpi p}\left(\frac{\mathfrak{h}-\mathfrak{g}}{2}-(\mathfrak{h}-\varrho)\right)^{p}\right]d\varrho$$

$$\leq \int_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}} \left[(\mathfrak{h}-\varrho)^{\varpi p}\left(\frac{\mathfrak{h}-\mathfrak{g}}{2}\right)^{p} d\varrho - \int_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}} (\mathfrak{h}-\varrho)^{\varpi p}(\mathfrak{h}-\varrho)^{p}\right] d\varrho$$

$$= \left(\frac{\mathfrak{h}-\mathfrak{g}}{2}\right)^{(\varpi+1)p+1}\left(\frac{p}{(\varpi p+1)\left[(\varpi+1)p+1\right]}\right),$$

$$\int_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}} \left[\left(\varrho-\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)^{p}\left[(\mathfrak{h}-\mathfrak{g})^{\varpi}-(\varrho-\mathfrak{g})^{\varpi}\right]^{p}\right] d\varrho$$

$$\leq \int_{\frac{\mathfrak{g}+\mathfrak{h}}{2}}^{\mathfrak{h}} \left[(\varrho-\mathfrak{g})^{p}-\left(\frac{\mathfrak{h}-\mathfrak{g}}{2}\right)^{p}\right]\left[(\mathfrak{h}-\mathfrak{g})^{\varpi p}-(\varrho-\mathfrak{g})^{\varpi p}\right] d\varrho$$

$$= \frac{(2^{p+1}-1)(\mathfrak{h}-\mathfrak{g})^{(\varpi+1)p+1}}{2^{p+1}(p+1)} - \frac{(\mathfrak{h}-\mathfrak{g})^{(\varpi+1)p+1}}{2^{p+1}} - \frac{p(\mathfrak{h}-\mathfrak{g})^{(\varpi+1)p+1}}{2^{(\varpi+1)p+1}\left[(\varpi+1)p+1\right](\varpi p+1)}$$

$$-\frac{(\mathfrak{h}-\mathfrak{g})^{(\varpi+1)p+1}}{(\varpi+1)p+1} + \frac{(\mathfrak{h}-\mathfrak{g})^{(\varpi+1)p+1}}{2^{p}(\varpi p+1)}$$

and

$$\left(\int_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}} \left[\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}-\varrho\right)^{p}\left[(\mathfrak{h}-\mathfrak{g})^{\varpi}-(\mathfrak{h}-\varrho)^{\varpi}\right]^{p}\right] d\varrho\right)^{\frac{1}{p}}$$

$$\leq \int_{\mathfrak{g}}^{\frac{\mathfrak{g}+\mathfrak{h}}{2}} \left[(\mathfrak{h}-\varrho)^{p}-\left(\frac{\mathfrak{h}-\mathfrak{g}}{2}\right)^{p}\right]\left[(\mathfrak{h}-\mathfrak{g})^{\varpi p}-(\mathfrak{h}-\varrho)^{\varpi p}\right] d\varrho$$

$$= \frac{(2^{p+1}-1)(\mathfrak{h}-\mathfrak{g})^{(\varpi+1)p+1}}{2^{p+1}(p+1)} - \frac{(\mathfrak{h}-\mathfrak{g})^{(\varpi+1)p+1}}{2^{p+1}} - \frac{p(\mathfrak{h}-\mathfrak{g})^{(\varpi+1)p+1}}{2^{(\varpi+1)p+1}\left[(\varpi+1)p+1\right](\varpi p+1)}$$

$$-\frac{(\mathfrak{h}-\mathfrak{g})^{(\varpi+1)p+1}}{(\varpi+1)p+1} + \frac{(\mathfrak{h}-\mathfrak{g})^{(\varpi+1)p+1}}{2^{p}(\varpi p+1)}.$$

Thus, by inserting all the calculated integrals into inequality (2.15), the desired result is achieved. □

Corollary 2.3. *The approach of ϖ towards 1 in Theorem 2.3 leads to the following result:*

$$\left|\frac{1}{2}\left[\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)+\frac{\Xi(\mathfrak{g})+\Xi(\mathfrak{h})}{2}\right]-\frac{1}{\mathfrak{h}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{h}}\Xi(\varrho)d\varrho\right|$$

$$\leq \left[\mathfrak{g}_1(p,q)+\mathfrak{h}_1(p,q)\right]$$

$$\times(\mathfrak{h}-\mathfrak{g})^2\left(|\Xi''(\mathfrak{g})|+2\left|\Xi''\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)\right|+|\Xi''(\mathfrak{h})|\right).$$

Example 2.2. Take the function $\Xi(\varrho)=\varrho^3$ over $[0,2]$. Then, the right-hand side of the inequality (2.14) is given by:

$$\begin{aligned}&2^{\varpi+4}3\times\Bigg[\frac{p^{\frac{1}{p}}}{2^{4-\frac{1}{p}+\varpi}(\varpi p+1)^{\frac{1}{p}}\left[(\varpi+1)p+1\right]^{\frac{1}{p}}}\\&+\frac{1}{2^{3-\frac{2}{p}}}\left(\frac{2^{p+1}-1}{2^{p+1}(p+1)}-\frac{1}{2^{p+1}}-\frac{p}{2^{(\varpi+1)p+1}\left[(\varpi+1)p+1\right](\varpi p+1)}\right.\\&\left.-\frac{1}{(\varpi+1)p+1}+\frac{1}{2^p(\varpi p+1)}\right)^{\frac{1}{p}}\Bigg]:=\Omega_1.\end{aligned}$$

On the other hand, the left-hand side of the inequality (2.14) becomes

$$\left|2^{\varpi-2}+\varpi 2^{\varpi-1}-\frac{2^{\varpi}(\varpi^2+2)}{\varpi+2}+\frac{2^{\varpi-1}(\varpi^2-2\varpi+3)}{\varpi+1}\right|:=\Omega_2.$$

Thus, according to Figure 2, for $p>1$ and $\varpi>1$, the left-hand side of inequality (2.14) is always lower than the right-hand side.

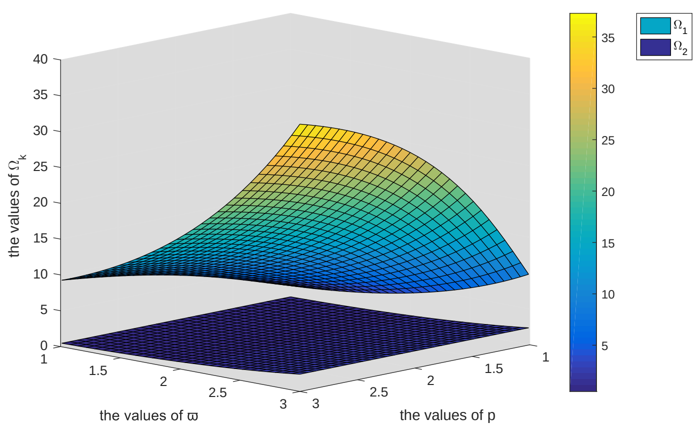

Figure 2. The graph of both sides of inequality (2.14) based on Example 2.2, calculated and plotted using MATLAB, is shown for $\varpi\in(1,3)$ and $p\in(1,3)$.

3. Applications

Now, we explore certain applications related to special means through positive real numbers. Before that, we review several well-known notions:

$$A(\mathfrak{g},\mathfrak{h})=\frac{\mathfrak{g}+\mathfrak{h}}{2},\ \text{(Arithmetic mean);}$$

$$K(\mathfrak{g},\mathfrak{h}) = \sqrt{\frac{\mathfrak{g}^2+\mathfrak{h}^2}{2}},\ \text{(Quadratic mean)};$$

$$G(\mathfrak{g},\mathfrak{h}) = \sqrt{\mathfrak{g}\mathfrak{h}},\ \text{(Geometric mean)};$$

$$I(\mathfrak{g},\mathfrak{h}) = \frac{1}{e}\left(\frac{\mathfrak{h}^{\mathfrak{h}}}{\mathfrak{g}^{\mathfrak{g}}}\right)^{\frac{1}{\mathfrak{h}-\mathfrak{g}}},\ \text{(Identric mean)};$$

$$L_n(\mathfrak{g},\mathfrak{h}) = \left(\frac{\mathfrak{h}^{n+1}-\mathfrak{g}^{n+1}}{(\mathfrak{h}-\mathfrak{g})(n+1)}\right)^{\frac{1}{n}},n\in\mathbb{R}\backslash\{-1,0\},\ \text{(Generalized log-mean)}.$$

Proposition 3.1. *Consider $n > 3$ and $\mathfrak{h} > \mathfrak{g} > 0$. Then,*

$$\left|\frac{A^n(\mathfrak{g},\mathfrak{h})}{2}+\frac{A(\mathfrak{g}^n,\mathfrak{h}^n)}{2}-L_n^n(\mathfrak{g},\mathfrak{h})\right| \le \frac{n(n-1)(\mathfrak{h}-\mathfrak{g})^2}{48}A(\mathfrak{g}^{n-2},\mathfrak{h}^{n-2}).$$

Proof. Let's examine the function $\Xi(\varrho) = \varrho^n$ with $\varrho > 0$. Then, we get $|\Xi''(\varrho)| = n(n-1)\varrho^{n-2}$. From the fact that $(|\Xi''(\varrho)|)'' = n(n-1)(n-2)(n-3)\varrho^{n-4} > 0$ for all $n > 3$, it follows that the function $|\Xi''|$ is convex for all $n > 3$. In this case, applying the function Ξ to Theorem 2.2, we obtain

$$\left|\frac{(\frac{\mathfrak{g}+\mathfrak{h}}{2})^n}{2}+\frac{\mathfrak{g}^n+\mathfrak{h}^n}{4}-\frac{\mathfrak{h}^{n+1}-\mathfrak{g}^{n+1}}{(\mathfrak{h}-\mathfrak{g})(n+1)}\right| \tag{3.1}$$
$$\le \frac{(\mathfrak{h}-\mathfrak{g})^2}{96}\left[n(n-1)\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right)^{n-2}+\frac{n(n-1)(\mathfrak{g}^{n-2}+\mathfrak{h}^{n-2})}{2}\right].$$

Hence, rearranging the inequality (3.1) yields the required result. □

Proposition 3.2. *Let $\mathfrak{h} > \mathfrak{g} > 0$. Then, the following inequality is true:*

$$\left|\frac{\ln A(\mathfrak{g},\mathfrak{h})+A(\ln\mathfrak{g},\ln\mathfrak{h})}{2}-\ln I(\mathfrak{g},\mathfrak{h})\right| \le \frac{(\mathfrak{h}-\mathfrak{g})^2}{48}\frac{K^2(\mathfrak{g},\mathfrak{h})}{G^4(\mathfrak{g},\mathfrak{h})}.$$

Proof. Let's analyze the function $\Xi(\varrho) = \ln\varrho$ with $\varrho > 0$. Because $|\Xi''(\varrho)| = \frac{1}{\varrho^2}$, we find that $|\Xi''|$ is a convex function on $[\mathfrak{g},\mathfrak{h}]$. When we utilize the function Ξ in Theorem 2.2, the result is

$$\left|\frac{\ln(\frac{\mathfrak{g}+\mathfrak{h}}{2})+\frac{\ln\mathfrak{g}+\ln\mathfrak{h}}{2}}{2}-\frac{\ln(\frac{\mathfrak{h}^{\mathfrak{h}}}{\mathfrak{g}^{\mathfrak{g}}})}{\mathfrak{h}-\mathfrak{g}}+1\right| \le \frac{(\mathfrak{h}-\mathfrak{g})^2}{48}\frac{\left(\frac{1}{\mathfrak{g}^2}+\frac{1}{\mathfrak{h}^2}\right)}{2}. \tag{3.2}$$

The desired inequality can be obtained by modifying the inequality (3.2). □

4. Conclusion

In the present paper, we introduce a new extension of classical Bullen-type inequalities involving the Riemann-Liouville fractional integrals. Then, we prove a fractional integral identity for the case of twice differentiable mappings and also present some significant inequalities by taking advantage of the convexity and the Hölder inequality. Moreover, we prove the correctness of our main findings using specific examples enriched by graphical representations. Next, we utilize these inequality formulas for the special means of a pair of positive integers. Lastly, we investigate some well-known results in the literature by making particular selections. In future studies, we hope that readers will be motivated to look into this subject further by our approaches and findings. Therefore, interested readers can investigate analogous inequalities for other fractional integrals and develop novel Bullen-type inequalities by employing different types of convexity.

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Conflict interests

The authors declare that they have no conflict interests.

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