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Bullen-Type Fractional Integral Inequalities Involving Twice-Differentiable Mappings with Applications

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Abstract In this paper, we first introduce a novel extension of the Bullen-type inequalities using Riemann-Liouville integral operators and establish a generalized fractional integral identity of the Bullen type. Next, we derive integral inequalities related to the Bullen-type inequalities for twice-differentiable mappings, utilizing this novel extended identity. To demonstrate these inequalities, we provide examples along with corresponding graphs. Furthermore, we investigate how these inequalities can be practically applied to mean inequalities, contributing to a deeper comprehension and wider usefulness of these mathematical ideas. Finally, the inequalities given in this work extend several known inequalities in the literature, which is the main advantage of the newly found inequalities.

Keywords Bullen-type inequalities, Hermite-Hadamard-type inequalities, c-onvex functions, Riemann-Liouville fractional integrals, special means

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1. Introduction

The theory of convex functions, which is a mathematical discipline, has been widely applied in various fields including optimization theory, control theory, energy systems, information theory and physics. Furthermore, convexity theory and its applications have offered solutions to numerous problems arising from various areas of mathematics. In particular, the emergence of many basic inequalities from convex functions has led to the rapid advancement of these functions. Besides, the application of integral inequalities satisfying the idea of convexity has been broadly utilized to achieve many novel outcomes in the theory of inequalities. The first essential outcome for a convex function is known as the Hermite-Hadamard inequality, which was investigated by C. Hermite and J. Hadamard [14, 21] stated as follows:

$$\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right) \leq \frac{1}{\mathfrak{h}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{h}} \Xi(\pi)d\pi \leq \frac{\Xi(\mathfrak{g})+\Xi(\mathfrak{h})}{2}, \tag{1.1}$$

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where $\Xi : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\mathfrak{g}, \mathfrak{h} \in I$ with $\mathfrak{g} < \mathfrak{h}$. If Ξ is a concave function, then two inequalities in the expression apply in the opposite direction.

For the average value of a convex function across a compact interval, the Hermite-Hadamard inequality gives both upper and lower bounds. Accordingly, this inequality is utilized in many branches of mathematics such as integral calculus, probability theory, statistics, and optimization, and also acts as a foundation for numerous other inequalities. Additionally, it is used to solve real-world problems in physics, engineering, economics, and other disciplines. The applications of this inequality keep on developing as new problems arise, making it an indispensable tool for resolving complex mathematical problems as well as issues from various fields of study. On the other hand, the Hermite-Hadamard inequality is identified by the trapezoid inequality on its right part and the midpoint inequality on its left part. Dragomir and Agarwal [13] initially demonstrated trapezoid-type inequalities for the case of convex functions, while Kırmacı [35] first established midpoint-type inequalities for the case of convex functions. Numerous studies have been conducted in this field since these inequalities emerged [2,31,42,43].

In [7] Bullen proved an alternative Hermite-Hadamard type inequality, known as Bullen's inequality: If $\Xi : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\mathfrak{g}, \mathfrak{h} \in I$ with $\mathfrak{g} < \mathfrak{h}$, then

$$\frac{1}{\mathfrak{h}-\mathfrak{g}} \int_{\mathfrak{g}}^{\mathfrak{h}} \Xi(\pi) d\pi \leq \frac{1}{2}\left[\Xi\left(\frac{\mathfrak{g}+\mathfrak{h}}{2}\right) + \frac{\Xi(\mathfrak{g}) + \Xi(\mathfrak{h})}{2}\right]. \tag{1.2}$$

The most important objective of these kinds of inequalities is that they provide more accurate and stronger information about error estimates of the widely examined quadrature and cubature rules. The inequality (1.2) also gives the error bounds for the remainder of Bullen quadrature schemes. So, researchers have focused their studies on these types of inequalities. For instance, Çakmak [12] obtained various novel inequalities for differentiable functions based on h-convexity including Bullen-type inequalities. Dragomir and Wang [15] established a natural extension of this inequalities. In [30], İşcan et al. presented a new general identity for differentiable functions and therefore obtained certain novel inequalities, including those of the Hermite-Hadamard-type and Bullen-type. Tseng et al. [45] used Lipschitz functions to study a few Hadamard-type and Bullen-type inequalities and created several applications through the use of special means. Following the appearance of these inequalities, considerable research has been carried out in this field [37,47].

While classical derivatives and integrals solve many problems in science and technology, they can be inadequate in certain cases. Fractional calculus dealing with derivatives and integrals of arbitrary order presents new ways to tackle these problems. Then, this approach offers a foundation for exploring and understanding systems governed by fractional dynamics, enabling a more detailed mathematical description of complex phenomena, see [8, 9, 11, 24, 32, 36, 39, 41]. Hence, with the advent of fractional integrals and derivatives such as Riemann-Liouville [20, 34], Caputo-Fabrizio [10], Atangana-Baleanu [3], tempered [40] and conformable [33], this calculus has become more prominent and has been utilized in diverse fields of science and engineering. Besides, many researchers have used fractional integral operators to establish different kinds of integral inequalities.