Local Ground-State and Mountain Pass Solutions for a p-Kirchhoff Equation with Critical Exponent

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Abstract We study a Kirchhoff-type equation where the diffusion coefficient is non-locally affected, the nonlinear diffusion phenomenon is governed by the p-Laplace operator and the population supply presents critical growth. The energy functional associated with the equation is not bounded from below so that there is no global ground-state; however, we prove the existence of a positive local ground-state. We also prove that the equation has a positive solution of mountain pass type. The concentration-compactness principle is a main tool in our approach.

Keywords Integro-differential equation, p-Kirchhoff equation, critical exponent, mountain-pass solution, local ground-state solution

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1. Introduction

In the 1950s the theoretical study of spatial diffusion of biological populations with PDEs started with [27] by naively considering the individuals as non-living particles, i.e., assuming that their movement is random. This produced the equation

$$\partial_t u = \Delta u + \sigma(u), \qquad t \geq 0, x \in \Omega,$$

where $u = u(t,x)$ denotes the population density, $\Omega \subseteq \mathbb{R}^N$ is the habitat and $\sigma(u)$ denotes the population supply due to births and deaths. Similar to the case of the heat equation, the randomness assumption implies that the speed of propagation becomes infinite. Obviously, the population supply could also be time-dependent but, to simplify the presentation, we don't consider this situation. PDE approaches have some advantages over stochastic ones as PDEs allow the influence of spatial

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structure while probability frameworks are not so helpful to unveil ecological laws for the space use [24].

Actually, the migration of individuals is not random. For example, in some species, like arctic squirrels, the individuals migrate to avoid crowding, [5, 14]. So, an important modeling advance came in [15], where, using the tools of continuum mechanics, the equation

$$\partial_t u = \Delta \eta(u) + \sigma(u), \qquad t \geq 0, x \in \Omega, \tag{1.1}$$

was obtained, where η is a non-linear function such that $\eta'(0) = 0$ and $\eta'(s) > 0$ if $s > 0$. Equation (1.1) is parabolic but degenerates to a first-order equation when $u = 0$, causing a population, initially living in a bounded habitat, to spread out of it at a finite velocity.

The theoretical study of biological diffusion is nowadays far from considering individuals as non-living particles and it's dealing even with cognitive processes, [13, 24]. In this modeling context, there naturally appear situations where the velocity of dispersion is given by

$$v = -aI(u)\,\nabla u, \tag{1.2}$$

where $a > 0$ and the diffusion coefficient, $\check{d} = aI(u)$, is affected by non-local population information like

$$I(u) = \int_\Omega |u|^\theta dx \quad \text{or} \quad I(u) = \int_\Omega |\nabla u|^\theta dx,$$

corresponding, respectively, to total population, (see e.g. [9–11]), and total energy (see e.g. [1, 7, 19, 21, 22, 25, 35]); here $\theta \geq 1$. In the case of (1.2), a balance of population gives the integro-differential equation

$$\partial_t u = aI(u)\Delta u + \sigma(u), \qquad t \geq 0, x \in \Omega. \tag{1.3}$$

Remark 1.1. Let $p > 1$. Recall that the p-Laplace operator and the p-biharmonic operator, given respectively by $\Delta_p w = \mathrm{div}(|\nabla w|^{p-2}\nabla w)$ and $\Delta_p^2 w = \Delta(|\Delta|^{p-2}\Delta w)$, are quasilinear and, for $p = 2$, coincide with the Laplace operator and the biharmonic operator, respectively.

Since the difficulty to model biological situations just increases, the corresponding equations will certainly have to consider additional non-linear ingredients, [24, 35, 37], and could even become of higher order, as is the case with the modeling of physical phenomena, [26, 31, 32]. Then, both from the mathematical point of view and from the theoretical population modeling perspective, it's interesting to ask whether it's possible to achieve results for quasilinear models (see e.g. [36]) and the situation when the diffusion phenomenon is mainly governed by the p-Laplace operator or the p-biharmonic operator; see e.g. [19, 20, 22, 36]).

Remark 1.2. Let's recall that Kirchhoff's original equation [16],

$$\partial_{tt} u - \left(a + b \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \right) \Delta u = \sigma(x, u),$$

is a non-local wave equation that considers changes in lenght of a string that are produced by transverse vibrations. Its time-independent counterpart has been extensively studied under different conditions on σ; see e.g. [6, 8, 35, 37].