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Existence Results for a Generalized $p(x)$-Biharmonic Problem Type with No-Flux Boundary Condition

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Abstract This paper aims to study the existence and multiplicity of weak solutions for a problem involving a generalized $p(x)$-biharmonic operator with no flux boundary condition. By using the variational techniques and the theory of the variable exponent Lebesgue spaces, we obtain the existence of at least one nontrivial solution and at least n distinct pairs of nontrivial weak solutions to this problem, respectively.

Keywords No-flux boundary condition, $p(x)$-biharmonic problem type, variational methods

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1. Introduction

In this paper, we discuss the existence and multiplicity of solutions to the following problem:

$$\begin{cases} \Delta\ a(\Delta u) + z(x)|u|^{p(x)-2}u = \lambda f(x,u) + \theta b(x)|u|^{t(x)-2}u \text{ in } \ \Omega, \\ u = constant,\ \Delta u = 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega, \\ \int_{\partial\Omega} \frac{\partial}{\partial n} a(\Delta u) ds = 0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, with a Lipschitz boundary $\partial\Omega$; $a : \mathbb{R}^+ \to \mathbb{R}$ is a real continuous function which generalizes $|\xi|^{p(x)-2}\xi$; p and t in $C_+(\overline{\Omega})$ with

$$C_+(\overline{\Omega}) := \{r \in C(\overline{\Omega}) : \quad 1 < r^- := \min_{x\in\overline{\Omega}} r(x) \le r^+ := \max_{x\in\overline{\Omega}} r(x) < \infty\};$$

$f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function having subcritical growth such that f is odd with respect to the second variable; $z \in L^\infty(\Omega)$ and there exists $z_0 > 0$ such

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that $z(x) \geq z_0$ for all $x \in \Omega$; b is a weight function in a generalized Lebesgue spaces and λ, θ are paramaters.

Recently, the study of fourth-order partial differential equations and variational problems with variable exponents have been very successful and flourished. It has a lot of applications that were initiated to describe some materials including physics, nonlinear electrorheological fluids, image restoration, and elastic mechanics, (see, for example, [4] and [16]).

Indeed, in [8] the authors proved that the following problem:

$$\begin{cases} \Delta\left(|\Delta u|^{p(x)-2}\Delta u\right) + a(x)|u|^{p(x)-2}u = f(x,u) + \lambda g(x,u) \text{ in } \Omega, \\ u = 0 = \Delta u \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

admits at least three weak solutions.

While M. Boureanu in [3] considered a class of fourth order elliptic problem with variable exponent in the following problem:

$$\begin{cases} \Delta\left(|\Delta u|^{p(x)-2}\Delta u\right) + a(x)|u|^{p(x)-2}u = \lambda f(x,u) \text{ in } \Omega, \\ u = constant,\ \Delta u = 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \\ \int_{\partial\Omega} \frac{\partial}{\partial n}(|\Delta u|^{p(x)-2}\Delta u)ds = 0, \end{cases} \tag{1.3}$$

and proved that it has at least two nontrivial weak solutions.

There are several works concerning the fourth order elliptic problems with variable exponent (see, for example, [2], [10], [12], [13] and [15]). In analogy to the investigations made in [3] and [11], we treat the general $p(x)$-biharmonic operator $\Delta a(\Delta u)$ along with no flux boundary condition. We prove in the first place that for $t^- < p^-$ our problem has at least one nontrivial weak solution, and secondly, that for $p^+ < t^-$ the problem (1.1) admits at least n distinct pairs of nontrivial weak solutions.

Throughout this work, we assume that p is a log-Hölder continuous function in $\overline{\Omega}$, namely,

$$|p(x) - p(y)| \leq \frac{C}{|\log|x-y||},\ \forall x, y \in \overline{\Omega} \text{ with } 0 < |x-y| < \frac{1}{2},$$

and we state the following conditions:

$(\mathbf{A_1})$ $a : \mathbb{R}^+ \to \mathbb{R}$ is a real continuous function, strictly convex with $a(-\xi) = -a(\xi)$, and the mapping $A : \mathbb{R}^+ \to \mathbb{R}$ defined as the primitive of a that vanishes at 0 is :

$$A(t) = \int_0^t a(s)ds.$$

$(\mathbf{A_2})$ $|\xi|^{p(x)} \leq a(\xi).\ \xi \leq p(x)A(\xi)$ for a.e $x \in \Omega$ and all $\xi \in \mathbb{R}^+$.

$(\mathbf{B_1})$ $b \in L^{q(x)}(\Omega)$ such that $\frac{1}{p_2^*(x)} + \frac{1}{q(x)} \leq \frac{1}{t(x)}$ for all $x \in \overline{\Omega}$, where $q, t \in C_+(\overline{\Omega})$

$$\text{and}\quad p_2^*(x) := \begin{cases} \dfrac{Np(x)}{N-2p(x)} & \text{if } p(x) < \frac{N}{2}, \\ +\infty & \text{if } p(x) \geq \frac{N}{2}. \end{cases}$$

($\mathbf{B_2}$) There exists a ball B such that $B \subset \Omega$ and $b(x) > 0$ for a.e. $x \in B$.
($\mathbf{F_1}$) There exist functions r_j and a_j with $(j = 1, ..., m_0)$ such that,

$$\begin{cases} r_j \in C_+(\overline{\Omega}), \\ \inf_{x\in\Omega}[t(x) - r_j(x)] > 0, \end{cases}$$

and

$$\begin{cases} a_j \in L_+^{\frac{t(.)}{t(.)-r_j(.)}}\left(b^{-\frac{r_j}{t-r_j}}, \Omega\right) \quad (j = 1, ..., m_0), \\ \max\limits_{1\le j\le m_0} r_j^+ > p^-, \end{cases}$$

with

$$|f(x,\tau)| \le \sum_{j=1}^{m_0} a_j(x)|\tau|^{r_j(x)-1} \text{ a.e } x \in \Omega, \tau \in \mathbb{R}.$$

($\mathbf{F_2}$) There exists $t_0 > 0$ such that,

$$\int_B F(x, t_0)dx > 0,$$

where B is a ball given by ($\mathbf{B_2}$), and F represents the antiderivative of f.

The existence result for the $t^- < p^-$ case is stated as follows:

Theorem 1.1. *Assume that hypotheses* ($\mathbf{A_1}$), ($\mathbf{A_2}$), ($\mathbf{F_1}$) *and* ($\mathbf{F_2}$) *are fulfilled. If* $t^- < p^-$*, there exists a constant* $\lambda_0 > 0$ *such that for every* $\lambda > \lambda_0$*, the problem* (1.1) *has at least one nontrivial weak solution.*

Next, we study the problem (1.1) with $p^+ < t^-$. In this case, we need the following additional assumptions.
($\mathbf{P}$) There exists $\omega \in L_+^{\frac{p^*(.)}{p^*(.)-p(.)}}$, such that

$$\lambda_* := \inf_{\varphi \in C_c^\infty(\Omega)} \frac{\int_\Omega |\Delta\varphi|^{p(x)}dx}{\int_\Omega \omega(x)|\varphi|^{p(x)}dx} \in (0, \infty).$$

The condition ($\mathbf{P}$) holds in [1]. Furthermore, we replace the condition (F_2) by the following conditions:
($\mathbf{F_2'}$) There exists an open ball $B' \subset \Omega$ and a function $m \in L_+^1(B')$ such that,

$$\lim_{|\tau|\to\infty} \frac{F(x,\tau)}{m(x)|\tau|^{t^+}} = \infty \text{ uniformly for a.e } x \in B'.$$

($\mathbf{F_3}$) For ω given by ($\mathbf{P}$), there exist $\alpha \in [p^+, t^-)$ and a bounded function $e \in L_+^1(\Omega) := \{u \in L^1(\Omega) : u > 0 \text{ a.e in } \Omega\}$ such that,

$$\alpha F(x,\tau) - \tau f(x,\tau) \le \frac{\alpha - p^+}{p^+}\omega(x)|\tau|^{p(x)} + e(x) \text{ for a.e. } x \in \Omega \text{ and all } \tau \in \mathbb{R}.$$

Theorem 1.2. *Assume that hypotheses* ***(P)****,* ($\mathbf{A_1}$), ($\mathbf{A_2}$), ($\mathbf{F_1}$), ($\mathbf{F_2'}$) *and* ($\mathbf{F_3}$) *are fulfilled. For any* $\lambda \in (0, \lambda_*]$ *there exists a constant* $\theta_0 > 0$ *such that for any* $\theta \in (0, \theta_0)$*, if* $p^+ < t^- < t^+ < p_2^*(x)$ *for all* $x \in \overline{\Omega}$*, the problem* (1.1) *admits at least* n *distinct pairs of nontrivial weak solutions.*

This work is organized as follows: In Section 2, we start with the basic results and recall some definitions and properties of Lebesgue-Sobolev spaces with variable exponents, as well as some properties that we will use later. Section 3 is devoted to the existence of solutions, where we cite our two main results concerning nontrivial weak solution for the first case and the multiplicity of the second.

2. Preliminaries

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Denote

$$h^+ := \sup_{x\in\overline{\Omega}} h(x) \quad \text{and} \quad h^- := \inf_{x\in\overline{\Omega}} h(x).$$

For $p \in C_+(\overline{\Omega})$ and a Lebesgue measurable and positive a.e. function $\omega : \Omega \to \mathbb{R}^+$, define the weighted variable exponent Lebesgue space by

$$L^{p(x)}(\omega,\Omega) := \left\{ u : \Omega \to \mathbb{R} \ \text{ mesurable}, \quad \int_\Omega \omega(x)|u(x)|^{p(x)}\mathrm{d}x < \infty \right\},$$

endowed with the Luxemburg norm:

$$\|u\|_{L^{p(x)}(\omega,\Omega)} := \inf\left\{ \lambda > 0 : \int_\Omega \omega(x) \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \le 1 \right\}.$$

When $w \equiv 1$, we write $L^{p(x)}(\Omega), \|u\|_{L^{p(x)}(\Omega)}$ in place of $L^{p(x)}(\omega,\Omega), \|u\|_{L^{p(x)}(\omega,\Omega)}$. $\left(L^{p(x)}(\Omega), \|u\|_{L^{p(x)}(\Omega)}\right)$ is a separable and reflexive Banach space, [14].

Then the following properties holds.

Proposition 2.1. *[6] Define $\rho : L^{p(x)}(\omega,\Omega) \to \mathbb{R}$ as:*

$$\rho(u) := \int_\Omega \omega(x)|u(x)|^{p(x)}dx, \ \forall u \in L^{p(x)}(\omega,\Omega).$$

Then, for all $u \in L^{p(x)}(\omega,\Omega)$ we have:
(i) $\|u\|_{L^{p(x)}(\omega,\Omega)} < 1 (resp := 1, > 1)$ if and only if $\rho(u) < 1 (resp := 1, > 1)$,
(ii) if $\|u\|_{L^{p(\cdot)}(\omega,\Omega)} > 1$, then $\|u\|^{p^-}_{L^{p(x)}(\omega,\Omega)} \le \rho(u) \le \|u\|^{p^+}_{L^{p(x)}(\omega,\Omega)}$,
(iii) if $\|u\|_{L^{p(\cdot)}(\omega,\Omega)} < 1$, then $\|u\|^{p^+}_{L^{p(x)}(\omega,\Omega)} \le \rho(u) \le \|u\|^{p^-}_{L^{p(x)}(\omega,\Omega)}$.

Proposition 2.2. *[14] For $u \in L^{p(x)}(\Omega)$ and $v \in L^{p_0(x)}(\Omega)$, we have the hölder-type inequality:*

$$\left|\int_\Omega u(x)v(x)dx\right| \le 2|u|_{p(x)}|v|_{p_0(x)},$$

where $L^{p_0(x)}$ is a conjugate space of $L^{p(x)}$.

Proposition 2.3. *[7] Let p and q be measurable functions such that $p \in L^\infty(\Omega)$ and $1 < p(x)q(x) < \infty$ for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$ such that $u \neq 0$. Then:*

1. $|u|_{p(x)q(x)} \le 1 \ \Rightarrow |u|^{p^+}_{p(x)q(x)} \le ||u|^{p(x)}|_{q(x)} \le |u|^{p^-}_{p(x)q(x)}$,
2. $|u|_{p(x)q(x)} \ge 1 \ \Rightarrow |u|^{p^-}_{p(x)q(x)} \le ||u|^{p(x)}|_{q(x)} \le |u|^{p^+}_{p(x)q(x)}$.

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^{\alpha}u \in L^{p(x)}(\Omega). |\alpha| \leq k\},$$

where $D^{\alpha}u := \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}...\partial x_N^{\alpha_N}}$, with $\alpha = (\alpha_1, ..., \alpha_N)$ a multi-index and $|\alpha| = \alpha_1 + ... + \alpha_N$. The space $W^{k,p(.)}(\Omega)$ is endowed with the norm

$$\|u\|_{W^{k,p(x)}(\Omega)} := \sum_{|\alpha|\leq k} \|D^{\alpha}u\|_{p(x)}.$$

It also becomes a separable and reflexive Banach space [14].

Denote, for $k \geq 1$,

$$p_k^*(x) := \begin{cases} \dfrac{Np(x)}{N - kp(x)} & \text{if } \ p(x) < \frac{N}{k}, \\ +\infty & \text{if } p(x) \geq \frac{N}{k}. \end{cases}$$

Proposition 2.4. *[9] Letting $p, r \in C_+(\overline{\Omega})$, we have the following assertions:*
i) *If $r(x) \leq p_k^*(x)$ there is a continuous embedding, $W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$.*
ii) *If $r(x) < p_k^*(x)$ there is a the compact embedding, $W^{k,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$.*

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_c^{\infty}(\Omega)$ in $W^{k,p(.)}(\Omega)$. Considering the log-Hölder continuity of the exponent $p(x)$, we find

$$W_0^{1,p(x)} = \{u \in W^{1,p(x)} : u = 0 \text{ on } \partial\Omega\},$$

endowed with the norm:

$$\|u\|_{W_0^{1,p(x)}(\Omega)} := \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

Proposition 2.5. *[9] There exists a positive constant C such that,*

$$\|u\|_{L^{p(x)}(\Omega)} \leq C\|\nabla u\|_{L^{p(x)}(\Omega)}, \ \forall u \in W_0^{1,p(x)}(\Omega).$$

Due to [[9], Proposition 2.1] We note that space $(W_0^{1,p(x)}, \|.\|_{W_0^{1,p(x)}(\Omega)})$, is a separable, reflexive, and Banach space.

Remark 2.1. The space $(W^{2,p(x)} \cap W_0^{1,p(x)})$ endowed with the norm

$$\begin{aligned} \|u\|_{W^{2,p(x)}\cap W_0^{1,p(x)}} &= \|u\|_{W^{2,p(x)}} + \|u\|_{W_0^{1,p(x)}} \\ &= \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)} + \sum_{|\alpha|=2} \|D^{\alpha}u\|_{L^{p(x)}(\Omega)} \end{aligned}$$

is a separable and reflexive Banach space.

In addition, according to [[18], Theorem 4.4], the norm $\|.\|_{W^{2,p(x)}\cap W_0^{1,p(x)}}$ and the norm $\|.\|_{L^{p(x)}(\Omega)}$ are equivalent on $(W^{2,p(x)} \cap W_0^{1,p(x)})$.

Throughout this work, we consider the space $\mathbf{X}$ defined by

$$\mathbf{X} = \{u \in W^{2,p(x)}(\Omega) : u|_{\partial\Omega} = constant\},$$

which can also be viewed as:

$$\mathbf{X} = \{u + c : u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega), c \in \mathbb{R}\}.$$

Let us choose on $\mathbf{X}$ with the norm $\|.\|$ defined by:

$$\|u\|_z := \inf\left\{\mu > 0 : \int_\Omega \left(\left|\frac{\Delta u(x)}{\mu}\right|^{p(x)} + z(x)\left|\frac{u(x)}{\mu}\right|^{p(x)}\right) \mathrm{d}x \leq 1\right\},$$

which represents a norm on both $W^{2,p(x)}(\Omega)$ and $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ and it is equivalent to the usual norm defined here (see, [8]).

Moreover, $\mathbf{X}$ is a closed subspace of the separable and reflexive Banach space $W^{2,p(x)}(\Omega)$ endowed with the usual norm, thus $(\mathbf{X}, \|.\|_{W^{2,p(x)}(\Omega)})$ is a separable and reflexive Banach space (see, [3]).

Proposition 2.6. *[3] Define $\Lambda : \mathbf{X} \to \mathbb{R}$ as:*

$$\Lambda(u) := \int_\Omega [|\Delta u|^{p(x)} + z(x)|u|^{p(x)}]\,dx.$$

Then, for all $u, u_n \in W^{2,p(x)}(\omega, \Omega)$ we have:
(i) if $\|u\|_z \geq 1$, then $\|u\|_z^{p^-} \leq \Lambda(u) \leq \|u\|_z^{p^+}$;
(ii) if $\|u\|_z \leq 1$, then $\|u\|_z^{p^+} \leq \Lambda(u) \leq \|u\|_z^{p^-}$;
(iii) if $\|u_n\|_z \to 0$ $(\to \infty)$, then $\Lambda(u_n) \to 0$ $(\to \infty)$.

Definition 2.1. The functional I is said to satisfy the Palais-Smale condition $(PS)_c$ at the level $c \in \mathbb{R}$ if, for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that

$$I(u_n) \to c \quad \text{in } \mathbb{R}, \quad \text{and} \quad I'(u_n) \to 0 \quad \text{in } X^*,$$

where X^* is the dual space of X, there exists a subsequence of $\{u_n\}$ that converges strongly in X to a critical point of I.

Using the same arguments as in [[11], Proposition 2.6] we can show the following proposition.

Proposition 2.7. *[11] Let $t \in C_+(\overline{\Omega}), b : \Omega \to \mathbb{R}$ be measurable and positive a.e. in Ω and $\sigma \in C(\overline{\Omega})$ such that $1 \leq \sigma(x) \leq t(x)$. Let $u_n \rightharpoonup u$ in $\mathbf{X}$ as $n \to \infty$, then:*

$$\int_\Omega m(x)\,|u_n|^{\sigma(x)}\,\mathrm{d}x \to \int_\Omega m(x)|u|^{\sigma(x)}\mathrm{d}x \quad \text{as} \quad n \to \infty,$$

$$\int_\Omega |m(x)|\,|u_n - u|^{\sigma(x)}\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty,$$

$$\int_\Omega |m(x)|\,|u_n|^{\sigma(x)-1}\,|u_n - u|\,\mathrm{d}x \to 0 \qquad \text{as} \quad n \to \infty.$$

Consequently, if $m \in L_+^{\frac{t(\cdot)}{t(\cdot)-\sigma(\cdot)}}\left(b^{-\frac{\sigma}{t-\sigma}}, \Omega\right)$, then:

$$\mathbf{X} \hookrightarrow\hookrightarrow L^{\sigma(\cdot)}(m, \Omega).$$

Proposition 2.8. *[17] Let $X = V \oplus W$, where X is a real Banach space and V is finite dimensional. Suppose that $I \in C^1(X, \mathbb{R})$ is an even functional satisfying $I(0) = 0$, and*

- $(\mathbf{I_1})$ *there exist constants $\rho, \beta > 0$, such that $I(u) \geq \beta$, for all $u \in \partial B_\rho \cap W$;*
- $(\mathbf{I_2})$ *there exists a subspace $\tilde{X}$ of X, and $L > 0$, such that $dimV < dim\tilde{X} < \infty$ and $\sup_{u\in\tilde{X}} I(u) < L$;*
- $(\mathbf{I_3})$ *considering L given by $(\mathbf{I_2})$, I satisfies the $(PS)_c$, for all $c \in [0.L]$.*

Then I possesses at least $dim\tilde{X} - dimV$ pairs of nontrivial critical points.

3. Existence results

By using Green's formula and taking into consideration the fact that $\mathbf{X}$ is closed subspace of $(W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega))$ together with the boundary conditions, and since p is log-Holder, then $C^\infty(\overline{\Omega})$ is dense in $W^{k,p(x)}(\Omega)$ (see, [5]). We deduce the following formulation:

Definition 3.1. We say that $u \in \mathbf{X}$ is a weak solution of the problem (1.1) if:

$$\int_\Omega a(\Delta u)\Delta v dx + \int_\Omega z(x)|u|^{p(x)-2}uv dx - \lambda\int_\Omega f(x,u)v dx - \theta\int_\Omega b(x)|u|^{t(x)-2}uv dx = 0,$$

for all $v \in \mathbf{X}$.

The functional associated with the problem (1.1) is given by:

$$I(u) = \int_\Omega (A(\Delta u) + \frac{z(x)}{p(x)}|u|^{p(x)})dx - \lambda\int_\Omega F(x,u)dx - \theta\int_\Omega \frac{b(x)}{t(x)}|u|^{t(x)}dx. \quad (3.1)$$

By the same steps as in [8], we can show the following proposition.

Proposition 3.1. *1. Let $D(u) = \int_\Omega A(\Delta u) + \frac{z(x)}{p(x)}|u|^{p(x)}dx$. Then the functional $D : \mathbf{X} \to \mathbb{R}$ is sequentially weakly lower semi continuous, $D \in C^1(\mathbf{X}, \mathbb{R})$.*

2. The derivative $D' : \mathbf{X} \to \mathbf{X}'$ is a strictly monotone, bounded homeomorphism, and is of type S_+, namely, $u_n \rightharpoonup u$, and $\limsup_{n\to\infty} D'(u_n)(u_n - u) \leq 0$ implies that $u_n \to u$, where $\to$ and $\rightharpoonup$ denote the strong and weak convergence, respectively.

Thus, I is of class C^1 and

$$\begin{aligned}\langle I'(u).v\rangle =& int_\Omega a(\Delta u)\Delta v \mathrm{d}x + \int_\Omega z(x)|u|^{p(x)-2}uv dx \\ &- \theta\int_\Omega b(x)|u|^{t(x)-2}uv\mathrm{d}x - \lambda\int_\Omega f(x,u)v dx.\end{aligned}$$

Then, a critical point of I is a weak solution to problem (1.1).

3.1. Proof of Theorem 1.1

We will prove Theorem 1.1 by showing that functional I is coercive and weakly lower semi-continuous.

In fact, we have for $\|u\| \geq 1$,

$$I(u) = \int_\Omega (A(\Delta u) + \frac{z(x)}{p(x)}|u|^{p(x)})dx - \lambda\int_\Omega F(x,u)dx - \theta\int_\Omega \frac{b(x)}{t(x)}|u|^{t(x)}dx.$$

$$\geq \int_\Omega (A(\Delta u) + \frac{z(x)}{p(x)}|u|^{p(x)})dx - \lambda \int_\Omega F(x,u)dx - \frac{\theta}{t^-}\int_\Omega b(x)|u|^{t(x)}dx.$$

From $(\mathbf{A_2})$ we obtain,

$$\int_\Omega (A(\Delta u) + \frac{z(x)}{p(x)}|u|^{p(x)})dx \geq \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + z(x)|u|^{p(x)})dx$$
$$\geq \frac{1}{p^+}\|u\|_z^{p^-}. \tag{3.2}$$

And by $(\mathbf{F_1})$ we have:

$$|f(x,s)| \leq \sum_{j=1}^{m_0} a_j(x)|s|^{r_j(x)-1} \text{ a.e } x \in \Omega, s \in \mathbb{R},$$

thus,

$$\int_\Omega F(x,u)\mathrm{d}x \leq \int_\Omega |\int_0^u f(x,s)\mathrm{d}s|dx,$$

$$\leq \sum_{j=1}^{m_0}\int_\Omega \frac{a_j(x)}{r_j(x)}|u|^{r_j(x)}dx. \tag{3.3}$$

By using Young's inequality, we obtain,

$$a_j(x)|u|^{r_j(x)} = a_j(x)b(x)^{\frac{-r_j(x)}{t(x)}} \times |u|^{r_j(x)}b(x)^{\frac{r_j(x)}{t(x)}}$$
$$\leq \frac{t(x)-r_j(x)}{t(x)}\left|a_j(x)b(x)^{\frac{-r_j(x)}{t(x)}}\right|^{\frac{t(x)}{t(x)-r_j(x)}} + \frac{r_j(x)}{t(x)}b(x)|u|^{t(x)},$$

hence,

$$\lambda\sum_{j=1}^{m_0}\int_\Omega \frac{a_j(x)}{r_j(x)}|u|^{r_j(x)}\mathrm{d}x \leq \lambda\sum_{j=1}^{m_0}\int_\Omega \frac{t(x)-r_j(x)}{r_j(x)t(x)}\left|a_j(x)b(x)^{\frac{-r_j(x)}{t(x)}}\right|^{\frac{t(x)}{t(x)-r_j(x)}}$$
$$+\frac{r_j(x)}{r_j(x)t(x)}b(x)|u|^{t(x)}\mathrm{d}x.$$

Since $\frac{t(x)-r_j(x)}{r_j(x)t(x)} \leq \frac{1}{r^-}$ and $\frac{r_j(x)}{r_j(x)t(x)} \leq \frac{1}{t^-}$, then,

$$\lambda\sum_{j=1}^{m_0}\int_\Omega \frac{a_j(x)}{r_j(x)}|u|^{r_j(x)}\mathrm{d}x \leq \frac{\lambda}{r^-}\sum_{j=1}^{m_0}\int_\Omega \left|a_j(x)b(x)^{\frac{-r_j(x)}{t(x)}}\right|^{\frac{t(x)}{t(x)-r_j(x)}}\mathrm{d}x$$
$$+\frac{\lambda}{t^-}\sum_{j=1}^{m_0}\int_\Omega b(x)|u|^{t(x)}\mathrm{d}x,$$
$$\leq \frac{\lambda}{r^-}\sum_{j=1}^{m_0}\int_\Omega b(x)^{-\frac{r_j(x)}{t(x)-r_j(x)}}|a_j(x)|^{\frac{t(x)}{t(x)-r_j(x)}}\mathrm{d}x$$

$$+\frac{\lambda m_0}{t^-}\int_\Omega b(x)|u|^{t(x)}\mathrm{d}x.$$

We have $a_j \in L_+^{\frac{t(x)}{t(x)-r_j(x)}}\left(b^{-\frac{r_j(x)}{t(x)-r_j(x)}},\Omega\right)$, therefore,

$$\lambda\sum_{j=1}^{m_0}\int_\Omega \frac{a_j(x)}{r_j(x)}|u|^{r_j(x)}\mathrm{d}x \le \frac{\lambda m_0 M}{r^-}+\frac{\lambda m_0}{t^-}\int_\Omega b(x)|u|^{t(x)}\mathrm{d}x, \tag{3.4}$$

where $M = \max_{1\le j\le m_0}\int_\Omega b(x)^{-\frac{r_j(x)}{t(x)-r_j(x)}}|a_j(x)|^{\frac{t(x)}{t(x)-r_j(x)}}\,\mathrm{d}x < \infty$. From (3.1), (3.2), (3.4), and by using Proposition 2.2, Proposition 2.3 we deduce

$$\begin{aligned}
I(u) &\ge \frac{1}{p^+}\|u\|_z^{p^-} - \frac{\lambda m_0 M}{r^-} - \frac{\lambda m_0}{t^-}\int_\Omega b(x)|u|^{t(x)}\mathrm{d}x - \frac{\theta}{t^-}\int_\Omega b(x)|u|^{t(x)}dx\\
&= \frac{1}{p^+}\|u\|_z^{p^-} - \frac{\lambda m_0 M}{r^-} - \frac{(\lambda m_0+\theta)}{t^-}\int_\Omega b(x)|u|^{t(x)}\mathrm{d}x\\
&\ge \frac{1}{p^+}\|u\|_z^{p^-} - \frac{\lambda m_0 M}{r^-} - 2\frac{(\lambda m_0+\theta)}{t^-}|b(x)|_{q(x)}||u|^{t(x)}|_{q_0(x)}\\
&\ge \frac{1}{p^+}\ \|u\|_z^{p^-} - \frac{\lambda m_0 M}{r^-} - 2\frac{(\lambda m_0+\theta)}{t^-}|b(x)|_{q(x)}\max\{|u|^{t^+}_{t(x)q_0(x)}, |u|^{t^-}_{t(x)q_0(x)}\}\\
&\ge \frac{1}{p^+}\|u\|_z^{p^-} - \frac{\lambda m_0 M}{r^-} - 2C\frac{(\lambda m_0+\theta)}{t^-}|b(x)|_{q(x)}\|u\|_z^{t^-}.
\end{aligned}$$

Since $t^- < p^-$, then I is coercive. As the functions $u \to \lambda\int_\Omega F(x,u)dx$ and $u \to \theta\int_\Omega \dfrac{b(x)}{t(x)}|u|^{t(x)}dx$ are weakly lower semi-continuous and the functional $D : \mathbf{X} \to \mathbb{R}$ is sequentially weakly lower semi continuous and convex uniformly, we deduce that I is weakly lower semi-continuous. Therefore I has a global minimum point $u \in \mathbf{X}$, that is a weak solution to problem (1.1).

Furthermore, for all $\theta > 0$ and by hypothesis $(\mathbf{B_2})$ we have

$$I(u) \le \int_\Omega (A(\Delta u) + \frac{z(x)}{p(x)}|u|^{p(x)})dx - \lambda\int_\Omega F(x,u)dx.$$

Now by using the idea of [[3], proof Theorem 6], given the ball B as specified in hypothesis $(\mathbf{F_2})$, we can take $\epsilon > 0$ sufficiently small such that

$$\overline{B_\epsilon} = \overline{\{x\in\Omega|\ \ \mathrm{dist}(x,B)\le\epsilon\}} \subset \Omega.$$

Then, we can construct the function $u_\epsilon \in C_c^1$ defined by:

$$u_\epsilon := \begin{cases} t_0 \quad \text{when} \quad x\in B,\\ 0 \text{ when } x \in \Omega|B_\epsilon.\end{cases}$$

Hence,

$$I(u_\epsilon) \le \int_\Omega (A(\Delta u_\epsilon) + \frac{z(x)}{p(x)}|u_\epsilon|^{p(x)})dx - \lambda\int_B F(x,t_0)dx - \lambda\int_{B_\epsilon|B} F(x,u_\epsilon)dx.$$

We are able to fix ϵ_0 sufficiently small such that there exists a positive constant α_0 with the property that

$$I(u_{\epsilon_0}) \leq \int_\Omega (A(\Delta u_{\epsilon_0}) + \frac{z(x)}{p(x)}|u_{\epsilon_0}|^{p(x)})dx - \lambda\alpha_0 \int_B F(x,t_0)dx.$$

Now, by taking

$$\lambda_0 := \frac{\int_\Omega (A(\Delta u_{\epsilon_0}) + \frac{z(x)}{p(x)}|u_{\epsilon_0}|^{p(x)})dx}{\alpha_0 \int_B F(x,t_0)dx} > 0,$$

we deduce that $I(u_{\epsilon_0}) < 0$ for all $\lambda > \lambda_0$. By choosing $u = u_{\epsilon_0}$ we obtain that u is nontrivial for all $\lambda > \lambda_0$ because $I(0) = 0$, and we have completed our proof.

To prove Theorem 1.2 we need the following lemmas:

Lemma 3.1. *Let **(P)**, $(\mathbf{A_1})$, $(\mathbf{A_2})$, $(\mathbf{F_1})$ and $(\mathbf{F_3})$ hold. Then for $\lambda \in (0,\lambda_*]$ and $\theta > 0$ the functional I satisfies the $(PS)_c$ condition.*

Proof. Let $\lambda \in (0,\lambda_*]$ and $\theta > 0$ and let $\{u_n\}_{n=1}^\infty$ be a $(PS)_c$-sequence for I. We first claim that $\{u_n\}_{n=1}^\infty$ is bounded in $\mathbf{X}$; we have

$$\begin{aligned} I(u_n) - \frac{1}{\alpha}\langle I'(u_n), u_n\rangle = &\int_\Omega (A(\Delta u_n) + \frac{z(x)}{p(x)}|u|^{p(x)})dx - \lambda \int_\Omega F(x,u_n)dx \\ &- \theta \int_\Omega \frac{b(x)}{t(x)}|u_n|^{t(x)}dx - \frac{1}{\alpha}\int_\Omega a(\Delta u_n)\Delta u_n + z(x)|u_n|^{p(x)}dx \\ &+ \frac{\theta}{\alpha}\int_\Omega b(x)|u_n|^{t(x)}dx + \frac{\lambda}{\alpha}\int_\Omega f(x,u_n)u_n dx. \end{aligned}$$

From $(\mathbf{A_2})$ we obtain

$$\int_\Omega A(\Delta u_n)dx \geq \frac{1}{p^+}\int_\Omega a(\Delta u_n).\Delta u_n dx, \tag{3.5}$$

and since $t^- \leq t(x)$, then,

$$\begin{aligned} I(u_n) - \frac{1}{\alpha}\langle I'(u_n), u_n\rangle \geq &\left(\frac{1}{p^+} - \frac{1}{\alpha}\right)\int_\Omega a(\Delta u_n).\Delta u_n + z(x)|u_n|^{p(x)}dx \\ &+ \theta \int_\Omega \left(\frac{1}{\alpha} - \frac{1}{t^-}\right) b(x)|u_n|^{t(x)}dx \\ &- \frac{\lambda}{\alpha}\int_\Omega \left[\alpha F(x,u_n) - f(x,u_n)u_n\right]dx. \end{aligned}$$

By $(\mathbf{F_3})$ we obtain:

$$\begin{aligned} \int_\Omega \left[\alpha F(x,u_n) - f(x,u_n)u_n\right]dx &\leq \int_\Omega (\frac{\alpha - p^+}{p^+})\omega(x)|u_n|^{p(x)}dx + \int_\Omega e(x)dx \\ &\leq (\frac{\alpha - p^+}{p^+})\int_\Omega \omega(x)|u_n|^{p(x)}\mathrm{d}x + \|e\|_{L^1(\Omega)}. \end{aligned}$$

Thus,

$$-\frac{\lambda}{\alpha}\int_\Omega \left[\alpha F(x,u_n) - f(x,u_n)u_n\right]\mathrm{d}x \geq -\frac{\lambda}{\alpha}\frac{(\alpha - p^+)}{p^+}\int_\Omega \omega(x)|u_n|^{p(x)}\mathrm{d}x - \frac{\lambda\|e\|_{L^1(\Omega)}}{\alpha}. \tag{3.6}$$

From **(P)** it holds that:

$$\frac{(\alpha-p^{+})\ \lambda_{*}}{p^{+}}\int_{\Omega}\omega(x)|u_n|^{p(x)}dx\leq\frac{(\alpha-p^{+})}{p^{+}}\int_{\Omega}|\Delta u_n|^{p(x)}dx,$$

then,

$$-\frac{\lambda}{\alpha\lambda_{*}}\frac{(\alpha-p^{+})\lambda_{*}}{p^{+}}\int_{\Omega}\omega(x)|u_n|^{p(x)}\mathrm{d}x\geq-\frac{\lambda}{\alpha\lambda_{*}}\frac{(\alpha-p^{+})}{p^{+}}\int_{\Omega}|\Delta u_n|^{p(x)}\mathrm{d}x,$$

hence,

$$-\frac{\lambda}{\alpha}\frac{(\alpha-p^{+})}{p^{+}}\int_{\Omega}\omega(x)|u_n|^{p(x)}\mathrm{d}x\geq-\frac{\lambda}{\alpha\lambda_{*}}\frac{(\alpha-p^{+})}{p^{+}}\int_{\Omega}|\Delta u_n|^{p(x)}\mathrm{d}x. \tag{3.7}$$

Using (3.5), (3.6), (3.7) and $(\mathbf{A_2})$ we find:

$$\begin{aligned}I(u_n)-\frac{1}{\alpha}\langle I'(u_n),u_n\rangle\geq&\left(\frac{1}{p^{+}}-\frac{1}{\alpha}\right)\int_{\Omega}|\Delta u_n|^{p(x)}+z(x)|u_n|^{p(x)}dx\\&-\frac{\lambda}{\alpha\lambda_{*}}\frac{(\alpha-p^{+})}{p^{+}}\int_{\Omega}|\Delta u_n|^{p(x)}+z(x)|u_n|^{p(x)}dx\\&-\frac{\lambda\|e\|_{L^{1}(\Omega)}}{\alpha}+\theta\int_{\Omega}\left(\frac{1}{\alpha}-\frac{1}{t^{-}}\right)b(x)|u_n|^{t(x)}dx\\\geq&\frac{(\lambda_{*}-\lambda)(\alpha-p^{+})}{\lambda_{*}\alpha p^{+}}\int_{\Omega}|\Delta u_n|^{p(x)}+z(x)|u_n|^{p(x)}dx\\&-\frac{\lambda\|e\|_{L^{1}(\Omega)}}{\alpha}+\theta\int_{\Omega}\left(\frac{1}{\alpha}-\frac{1}{t^{-}}\right)b(x)|u_n|^{t(x)}dx.\end{aligned}$$

Recalling that $I(u_n)\to c$ and $I'(u_n)\to 0$, we deduce that for n large,

$$c+1+\|u_n\|_z\geq\frac{(\lambda_{*}-\lambda)(\alpha-p^{+})}{\lambda_{*}\alpha p^{+}}\|u_n\|_z^{p^{-}}-\frac{\lambda\|e\|_{L^{1}(\Omega)}}{\alpha}+\theta\int_{\Omega}\left(\frac{1}{\alpha}-\frac{1}{t^{-}}\right)b(x)|u_n|^{t(x)}\mathrm{d}x,$$

then,

$$\theta\left(\frac{1}{\alpha}-\frac{1}{t^{-}}\right)\int_{\Omega}b(x)|u_n|^{t(x)}\mathrm{d}x\leq c+1+\|u_n\|_z-\frac{(\lambda_{*}-\lambda)(\alpha-p^{+})}{\lambda_{*}\alpha p^{+}}\|u_n\|_z^{p^{-}}+\frac{\lambda\|e\|_{L^{1}(\Omega)}}{\alpha}.$$

Since $\frac{\lambda\|e\|_{L^{1}(\Omega)}}{\alpha}<\infty$, we obtain:

$$\int_{\Omega}b(x)|u_n|^{t(x)}\mathrm{d}x\ \leq R(1+\|u_n\|_z), \tag{3.8}$$

with R a positive constant independent of n. On the other hand we have,

$$\int_{\Omega}A(\Delta u_n)+\frac{z(x)}{p(x)}|u_n|^{p(x)}dx\leq I(u)+\lambda\int_{\Omega}F(x,u)dx+\theta\int_{\Omega}\frac{b(x)}{t(x)}|u|^{t(x)}dx,$$

therefore,

$$\frac{1}{p^{+}}\|u_n\|_z^{p^{-}}\leq c+1+\frac{\lambda m_0M}{r^{-}}+\frac{\theta+\lambda m_0}{t^{-}}\int_{\Omega}b(x)|u_n|^{t(x)}dx,$$

then,

$$\|u_n\|_z^{p^-} \leq p^+(c+1) + \frac{p^+\lambda m_0 M}{r^-} + \frac{p^+(\theta+\lambda m_0)}{t^-}\int_\Omega b(x)|u_n|^{t(x)}dx.$$

Since $r^- > 1$ and $t^- > p^+$ then,

$$\|u_n\|_z^{p^-} \leq p^+(c+1) + p^+\lambda m_0 M + (\theta+\lambda m_0)\int_\Omega b(x)|u_n|^{t(x)}dx,$$

hence,

$$\|u_n\|_z^{p^-} \leq D(1+\int_\Omega b(x)|u_n|^{t(x)}dx), \tag{3.9}$$

with K a positive constant independent of n.

From (3.8), (3.9) and since $p^- > 1$ we obtain the boundedness of $\{u_n\}_{n=1}^{\infty}$ in $\mathbf{X}$. Then, there exists a subsequence still denoted by (u_n) and $u \in \mathbf{X}$ such that $u_n \rightharpoonup u$ and $n \to +\infty$ so $\lim_{n\to+\infty}\langle I'(u_n), u_n - u\rangle = 0$, thus,

$$\lim_{n\to+\infty}(\int_\Omega a(\Delta u_n)\Delta(u_n-u) + z(x)|u_n|^{p(x)-2}u_n(u_n-u)dx - \lambda\int_\Omega f(x,u_n)(u_n-u)dx$$
$$-\theta\int_\Omega b(x)|u_n|^{t(x)-2}u_n(u_n-u)dx) = 0.$$

By $(\mathbf{F_1})$, we deduce

$$\int_\Omega f(x,u_n)(u_n-u)dx \leq \int_\Omega |\sum_{j=1}^{m_0} a_j(x)||u_n|^{r_j(x)-1}|(u_n-u)|dx.$$

Since $a_j \in L_+^{\frac{t(.)}{t(.)-r_j(.)}}\left(b^{-\frac{r_j}{t-r_j}},\Omega\right)$, we get,

$$\lim_{n\to+\infty}\int_\Omega f(x,u_n)(u_n-u)dx = 0.$$

By using Proposition 2.7, we easily obtain

$$\lim_{n\to+\infty}\int_\Omega b(x)|u_n|^{t(x)-2}u_n(u_n-u)dx = 0$$
$$\text{and}\quad \lim_{n\to+\infty}\int_\Omega z(x)|u_n|^{p(x)-2}u_n(u_n-u)dx = 0,$$

then, we arrive at

$$\lim_{n\to+\infty}\int_\Omega a(\Delta u_n)\Delta(u_n-u)dx = 0.$$

Hence, $u_n \to u$ in $\mathbf{X}$ due to the (S_+) property. In other words I satisfies the $(PS)_c$ condition and the proof is complete. □

Let $\mathbf{X}$ be a separable and reflexive Banach space [3]. Then there exist $\{e_j\} \subset \mathbf{X}$ and $\{e_j^*\} \subset \mathbf{X}^*$ such that:

$$X = \overline{\mathbf{span}}\{e_j : j = 1,2,...\}, \qquad X^* = \overline{\mathbf{span}}\{e_j^* : j = 1,2,...\},$$

with $\langle e_j, e_j^* \rangle = 1$ if $i = j$ and $\langle e_j, e_j^* \rangle = 0$ if $i \neq j$. Define:

$$\mathbf{X_j} = \text{span}\{e_j : j = 1, 2, ..\} \qquad \mathbf{V_k} = \bigoplus_{j=1}^{k} \mathbf{X_j} \qquad \mathbf{W_k} = \overline{\bigoplus_{j=k}^{\infty} \mathbf{X_j}}.$$

Then, $\mathbf{X} = \mathbf{V_k} \oplus \mathbf{W_k}$ and $\dim \mathbf{V_k} = k$.

Lemma 3.2. *Assume that* $(\mathbf{F_2'})$ *hold, and let* $\lambda, \theta > 0$. *Then,* $I(u) < 0$.

Proof. Let $\tilde{\mathbf{X}}$ from a sequence $\{\mathbf{X_k}\}_{k=1}^{\infty}$ of linear subspaces of $\mathbf{X}$, by $(\mathbf{F_2'})$ we find $T_k > 0$ such that

$$F(x, \tau) \geq \gamma_k m(x) |\tau|^{t^+}, \text{ a.e.} x \in B', \ \forall |\tau| \geq T_k,$$

where $\gamma_k > 0$ and B' given by $(\mathbf{F_2'})$. So

$$F(x, \tau) \geq \gamma_k m(x) |\tau|^{t^+} - \sup_{|\tau| \leq T_k} |F(x, \tau)|, \text{ a.e.} x \in B', \ \forall \tau \in \mathbb{R}. \tag{3.10}$$

On the other hand

$$\begin{aligned} I(u) &\leq \int_\Omega (A(\Delta u) + \frac{z(x)}{p(x)} |u|^{p(x)}) dx - \lambda \int_\Omega F(x, u) dx - \theta \int_\Omega \frac{b(x)}{t(x)} |u|^{t(x)} dx, \\ &\leq \int_\Omega (A(\Delta u) + \frac{z(x)}{p(x)} |u|^{p(x)}) dx - \lambda \int_{B'} F(x, u) dx - \theta \int_{B'} \frac{b(x)}{t(x)} |u|^{t(x)} dx. \end{aligned}$$

Thus,

$$\begin{aligned} I(u) \leq & \int_\Omega (A(\Delta u) + \frac{z(x)}{p(x)} |u|^{p(x)}) dx - \lambda \gamma_k ||u||_{L^{t^+}(m, B')}^{t^+} \\ & + \lambda \int_{B'} \sup_{|\tau| \leq T_k} |F(x, \tau)| dx - \theta \int_{B'} \frac{b(x)}{t(x)} |u|^{t(x)} dx \\ := & \tilde{I}(u). \end{aligned}$$

Taking $s > 1$ we obtain:

$$\begin{aligned} \tilde{I}(su) = & \int_\Omega (A(\Delta su) + \frac{z(x)}{p(x)} |su|^{p(x)}) dx - \lambda \gamma_k ||su||_{L^{t^+}(m, B')}^{t^+} \\ & + \lambda \int_{B'} \sup_{|\tau| \leq T_k} |F(x, \tau)| dx - \theta \int_{B'} \frac{b(x)}{t(x)} |su|^{t(x)} dx, \\ \leq & s^{p^+} \int_\Omega (p^- A(\Delta u) + z(x) |u|^{p(x)}) dx - s^{t^+} \lambda \gamma_k ||u||_{L^{t^+}(m, B')}^{t^+} \\ & + \lambda \int_{B'} \sup_{|\tau| \leq T_k} |F(x, \tau)| dx - \frac{s^{t^-} \theta}{t^+} \int_{B'} b(x) |u|^{t(x)} dx. \end{aligned}$$

Since $p^+ < t^-$, then $I(u) < 0$. The proof is achieved. □

Lemma 3.3. *[11] Define*

$$\delta_k := \sup_{v \in W_k, \|v\| \leq 1} \max_{1 \leq j \leq m_0} \|v\|_{L^{r_j(\cdot)}(a_j, \Omega)}.$$

The sequence $\{\delta_k\}_{k \geq 1}$ *above satisfies* $0 < \delta_{k+1} < \delta_k$, *for all* $k \in \mathbb{N}$, *and*

$$\lim_{k \to \infty} \delta_k = 0.$$

Lemma 3.4. *Assume that* $(\mathbf{A_2})$ *and* $(\mathbf{F_1})$ *hold. Then there exist constants* $\theta_0, \rho, \beta > 0$*, such that* $I(u) \geq \beta$*, for all* $u \in \partial B_\rho \cap W$*, and* $0 < \theta < \theta_0$*.*

Proof. From Lemma 3.3 we have

$$\forall \epsilon > 0, \exists k_0 \in \mathbb{N}, \forall k \geq k_0,\ \delta_k < \epsilon < 1,$$

then, we find $k_0 \in \mathbb{N}$ such that $0 < \delta_{k_0} < 1$. For $u \in W_{k_0}$, with $\|u\| \geq 1$ and by using $(\mathbf{A_2})$, $(\mathbf{F_1})$ we have

$$I(u) \geq \frac{1}{p^+}\|u\|_z^{p^-} - \lambda \sum_{j=1}^{m_0} \frac{1}{r_j^-}\|u\|_{L^{r_j(.)}(a_j,\Omega)}^{r_j^+} - \frac{2C|b(x)|_{q(x)}\theta}{t^-}\|u\|_z^{t^-}.$$

Furthermore, we consider $v = \frac{s}{\|s\|}$, with $s \in W_{k_0}\backslash\{0\}$, then,

$$\delta_{k_0} = \sup_{v \in W_k, \|v\| \leq 1} \max_{1 \leq j \leq m_0} \|v\|_{L^{r_j(.)}(a_j,\Omega)} \geq \max_{1 \leq j \leq m_0} \left\|\frac{s}{\|s\|}\right\|_{L^{r_j(.)}(a_j,\Omega)},$$

$$\left(\text{because } \left\|\frac{s}{\|s\|}\right\| = 1 \text{ and } \frac{s}{\|s\|} \in W_{k_0}\backslash\{0\}\right).$$

Hence, $\max_{1 \leq j \leq m_0} \|s\|_{L^{r_j(.)}(a_j,\Omega)} \leq \delta_{k_0}\|s\|_z$.

In addition we pose $r_* = \min\limits_{1 \leq j \leq m_0} r_j^-$, $r^* = \max\limits_{1 \leq j \leq m_0} r_j^+$, so we get,

$$I(u) \geq \frac{1}{p^+}\|u\|_z^{p^-} - \frac{\lambda m_0}{r_*}\delta_{k_0}^{r^*}\|u\|_z^{r^*} - \frac{2C|b(x)|_{q(x)}\theta}{t^-}\|u\|_z^{t^-}.$$

Let $\rho_{k_0} > 0$, such that,

$$\frac{m_0 \lambda \delta_{k_0}^{r^*}}{r_*}\rho_{k_0}^{r^*} = \frac{1}{2p^+}\rho_{k_0}^{p^-}$$

i.e

$$(\rho_{k_0})^{r^* - p^-} = \left(\frac{r_*}{2m_0\lambda p^+}\delta_{k_0}^{r^*}\right), \text{ with } r^* > p^-,$$

then

$$\rho_{k_0} = \left(\frac{r_*}{2m_0\lambda p^+}\delta_{k_0}^{r^*}\right)^{\frac{1}{r^* - p^-}}.$$

Thus

$$I(u) \geq \frac{1}{2p^+}\rho_{k_0}^{p^-} - \frac{2C\theta|b(x)|_{q(x)}}{t^-}\rho_{k_0}^{t^+},\ \forall u \in \partial B_{\rho_{k_0}} \cap W_{k_0}.$$

Therefore, by choosing $\mathbf{V} = \mathbf{V_{k_0}}$, $\mathbf{W} = \mathbf{W_{k_0}}$ and $\theta_0 = \frac{t^- \rho_{k_0}^{p^- - t^+}}{4p^+ C|b(x)|_{q(x)}} > 0$ we have that for any $\theta \in (0, \theta_0)$,

$$I(u) \geq \beta,\ \forall u \in \partial B_{\rho_{k_0}} \cap W,$$

with $\rho = \rho_{k_0}$ and $\beta = \frac{2C|b(x)|_{q(x)}}{t^-}\rho_{k_0}^{t^+}(\theta_0 - \theta) > 0$. □

3.2. Proof of Theorem 1.2

The functional I given by (3.1) is of class C^1, and a critical point of I is a weak solution to problem (1.1). Furthermore, I is even on $\mathbf{X}$ and $I(0) = 0$. Then from Lemma 3.1, Lemma 3.2, and Lemma 3.4, all conditions of Proposition 2.8 are satisfied. Hence I possesses at least $n = \dim\widetilde{X} - \dim V$ pairs of nontrivial critical points, moreover problem (1.1) has at least n pairs of nontrivial weak solutions, which completes the proof of Theorem 1.2.

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