Existence Results for a Generalized $p(x)$-Biharmonic Problem Type with No-Flux Boundary Condition

Youssef Zine.eddine[1,†], Mohamed Talbi[2], Najib Tsouli[1] and Mohammed Filali[1]

Abstract This paper aims to study the existence and multiplicity of weak solutions for a problem involving a generalized $p(x)$-biharmonic operator with no flux boundary condition. By using the variational techniques and the theory of the variable exponent Lebesgue spaces, we obtain the existence of at least one nontrivial solution and at least n distinct pairs of nontrivial weak solutions to this problem, respectively.

Keywords No-flux boundary condition, $p(x)$-biharmonic problem type, variational methods

MSC(2010) 35J60, 46E35, 35L20, 35J35.

1. Introduction

In this paper, we discuss the existence and multiplicity of solutions to the following problem:

$$\begin{cases} \Delta\, a(\Delta u) + z(x)|u|^{p(x)-2}u = \lambda f(x,u) + \theta b(x)|u|^{t(x)-2}u \text{ in } \Omega, \\ u = constant, \ \Delta u = 0 \qquad\qquad\qquad\qquad\quad \text{ on } \partial\Omega, \\ \int_{\partial\Omega} \frac{\partial}{\partial n} a(\Delta u)ds = 0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, with a Lipschitz boundary $\partial\Omega$; $a: \mathbb{R}^+ \to \mathbb{R}$ is a real continuous function which generalizes $|\xi|^{p(x)-2}\xi$; p and t in $C_+(\overline{\Omega})$ with

$$C_+(\overline{\Omega}) := \{r \in C(\overline{\Omega}): \ 1 < r^- := \min_{x\in\overline{\Omega}} r(x) \le r^+ := \max_{x\in\overline{\Omega}} r(x) < \infty\};$$

$f: \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function having subcritical growth such that f is odd with respect to the second variable; $z \in L^\infty(\Omega)$ and there exists $z_0 > 0$ such

---

[†]the corresponding author.
 Email address:ussefzineeddine@gmail.com(Y. Zine.eddine),talbijdi@gmail.com(M. Talbi), n.tsouli@ump.ac.ma(N. Tsouli),filali1959@yahoo.fr(M. Filali).
[1]Department of Mathematics, University Mohammed I, Faculty of sciences, 60000 Oujda, Morocco.
[2]CRMEF, 60000 Oujda, Morocco.

that $z(x) \geq z_0$ for all $x \in \Omega$; b is a weight function in a generalized Lebesgue spaces and λ, θ are paramaters.

Recently, the study of fourth-order partial differential equations and variational problems with variable exponents have been very successful and flourished. It has a lot of applications that were initiated to describe some materials including physics, nonlinear electrorheological fluids, image restoration, and elastic mechanics, (see, for example, [4] and [16]).

Indeed, in [8] the authors proved that the following problem:

$$
\begin{cases}
\Delta \left(|\Delta u|^{p(x)-2}\Delta u\right) + a(x)|u|^{p(x)-2}u = f(x,u) + \lambda g(x,u) \text{ in } \Omega, \\
u = 0 = \Delta u \hspace{6cm} \text{on } \partial\Omega,
\end{cases}
\tag{1.2}
$$

admits at least three weak solutions.

While M. Boureanu in [3] considered a class of fourth order elliptic problem with variable exponent in the following problem:

$$
\begin{cases}
\Delta \left(|\Delta u|^{p(x)-2}\Delta u\right) + a(x)|u|^{p(x)-2}u = \lambda f(x,u) \text{ in } \Omega, \\
u = constant, \ \Delta u = 0 \hspace{4cm} \text{on } \partial\Omega, \\
\int_{\partial\Omega} \frac{\partial}{\partial n}(|\Delta u|^{p(x)-2}\Delta u)ds = 0,
\end{cases}
\tag{1.3}
$$

and proved that it has at least two nontrivial weak solutions.

There are several works concerning the fourth order elliptic problems with variable exponent (see, for example, [2], [10], [12], [13] and [15]). In analogy to the investigations made in [3] and [11], we treat the general $p(x)$-biharmonic operator $\Delta a(\Delta u)$ along with no flux boundary condition. We prove in the first place that for $t^- < p^-$ our problem has at least one nontrivial weak solution, and secondly, that for $p^+ < t^-$ the problem (1.1) admits at least n distinct pairs of nontrivial weak solutions.

Throughout this work, we assume that p is a log-Hölder continuous function in $\overline{\Omega}$, namely,

$$
|p(x) - p(y)| \leq \frac{C}{|\log|x-y||}, \ \forall x, y \in \overline{\Omega} \text{ with } 0 < |x-y| < \frac{1}{2},
$$

and we state the following conditions:

($\mathbf{A_1}$) $a : \mathbb{R}^+ \to \mathbb{R}$ is a real continuous function, strictly convex with $a(-\xi) = -a(\xi)$, and the mapping $A : \mathbb{R}^+ \to \mathbb{R}$ defined as the primitive of a that vanishes at 0 is :

$$
A(t) = \int_0^t a(s)ds.
$$

($\mathbf{A_2}$) $|\xi|^{p(x)} \leq a(\xi). \xi \leq p(x)A(\xi)$ for a.e $x \in \Omega$ and all $\xi \in \mathbb{R}^+$.
($\mathbf{B_1}$) $b \in L^{q(x)}(\Omega)$ such that $\frac{1}{p_2^*(x)} + \frac{1}{q(x)} \leq \frac{1}{t(x)}$ for all $x \in \overline{\Omega}$, where $q, t \in C_+(\overline{\Omega})$

$$
\text{and } \ p_2^*(x) := \begin{cases}
\dfrac{Np(x)}{N-2p(x)} & \text{if } \ p(x) < \frac{N}{2}, \\
+\infty & \text{if } p(x) \geq \frac{N}{2}.
\end{cases}
$$